Q. How do you compose graphs $\Gamma_f \circ \Gamma_g = \Gamma_{f\circ g}$?

Complex Manifolds

Defn. A complex manifold is a manifold equipped with an equivalence class of complex atlases.

Defn. Two complex atlases of M are equivalent if their union is a complex atlas.

Defn. A complex atlas is a collection of complex charts on M whose domains cover M which are pairwise compatible.

Defn. A complex chart on M is a homeomorphism

$$\begin{array}{ccc} \varphi : \mathcal{U} & \longrightarrow & \Omega \\ \overset{\text{open}}{\cap} & & \cap \\ M & & \mathbb{C}^n \end{array}$$

for some n.

Defn. Complex charts $\varphi_i : \mathcal{U}_i \to \Omega_i$, $i=1,2$ are compatible if

$$\varphi_1^{-1} \circ \varphi_2 : \tilde{\Omega}_1 \subseteq \Omega_1 \longrightarrow \tilde{\Omega}_2 \subseteq \Omega_2$$

open subsets of $\mathbb{C}^n$

is biholomorphic.

Defn. φ is biholomorphic if both φ and φ^{-1} are holomorphic.

Defn. A map $f = (f_1, \ldots, f_m) : \Omega \underset{\text{open}}{\subseteq} \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is holomorphic if $\forall i, j$, $\forall z_k$, $k \neq i$

$$z_i \mapsto f_j : z_i \longmapsto f_j(z_1, \ldots, z_n)$$

is holomorphic as a map $\mathbb{C} \to \mathbb{C}$.

Lemma $f : \Omega \to \mathbb{C}^m$ is holomorphic
$\Leftrightarrow$ $\forall z \in \Omega$, $df|_z : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is complex linear.
i.e. $\boxed{df \circ j = J \circ df}$ where $j : \mathbb{C}^n \to \mathbb{C}^n$ and $J : \mathbb{C}^m \to \mathbb{C}^m$ are the complex structures on the respective vsp's.
(multiplication by $i = \sqrt{-1}$)

Note: The domain of df is $\operatorname{span}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}$ where $z_k = x_k + iy_k$. $j\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}$, $j\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}$ and similarly for J.

Proof. As we must do this for each coordinate, we omit indices:

Consider $f: \mathbb{C} \to \mathbb{C}$ and write $f = g + ih$

$$(df)\left(j \frac{\partial}{\partial x}\right) = J \circ df\left(\frac{\partial}{\partial x}\right)$$

$$\Leftrightarrow \quad df\left(\frac{\partial}{\partial y}\right) = J\left(\frac{\partial f}{\partial x}\right) = J\left(\frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}\right)$$

$$\Leftrightarrow \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} + i\frac{\partial h}{\partial y} = -\frac{\partial h}{\partial x} + i\frac{\partial g}{\partial x}$$

$$\Leftrightarrow \quad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial h}{\partial y} = \frac{\partial g}{\partial x}$$

This is the Cauchy Riemann condition for holomorphicity.

Apply a similar argument to $(df)\left(j\frac{\partial}{\partial y}\right) = J \circ df\left(\frac{\partial}{\partial y}\right)$ and one also arrives at CR.

Thus, $f = (f_1, \ldots, f_m): \Omega \subseteq_{\text{open}} \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic iff $\forall k, l$

$$df_k \circ j\left(\frac{\partial}{\partial x_l}\right) = J \circ df_k\left(\frac{\partial}{\partial x_l}\right) \quad \text{and} \quad df_k \circ j\left(\frac{\partial}{\partial y_l}\right) = J \circ df_k\left(\frac{\partial}{\partial y_l}\right)$$

By linearity over $\mathbb{R}$, this is equivalent to

$df_k \circ j = J \circ df_k$ on all of $T_z\mathbb{C}^n$ $\forall k$,

so $df \circ j = J \circ df$.

Corollary A complex manifold M is naturally almost complex.

Indeed: Given a chart $\varphi : \mathcal{U} \to \Omega$, this induces a complex structure on $d\varphi_q : T_q M \xrightarrow{\sim} \mathbb{C}^n$. The complex structure is independant of chart, since transition maps are complex isomorphisms.

Lemma A map between complex manifolds $M \xrightarrow{f} N$ is <u>holomorphic</u> if and only if $\forall q \in M$, $df : T_q M \to T_{f(q)} N$ is <u>complex linear</u>.

<u>Definition</u>: If $f : M \to N$ is a map between almost complex manifolds, f is called <u>(pseudo)holomorphic</u> if $df : T_q M \to T_{f(q)} N$ is complex linear for all $q \in M$.

E.g. $\dim = 2$ An almost complex structure is equivalent to an orientation <u>and</u> a conformal structure (an inner product up to scaling).

Fact In two dimensions a conformal structure yields local isothermal coordinates: a chart $\varphi = (x,y) : U \to \Omega \subseteq \mathbb{R}^2$ such that the conformal structure is standard: $(dx)^2 + (dy)^2$

"Betrami equation"

Corollary In $\dim_{\mathbb{R}} = 2$, every almost complex structure comes from a complex atlas.

Remark: Typically on an almost complex manifold (M, J) there exist many holomorphic curves (e.g. discs $\underset{\mathbb{C}}{\overset{\cap}{D^2}} \to M$), but rarely do there exist holomorphic functions $M \to \mathbb{C}$.

If near each point q a holomorphic chart $(f_1, \ldots, f_n) : U \to \mathbb{C}^n$ does exist, then the set of all such charts is a maximal complex atlas.

Defn. A **Kähler** manifold is a symplectic manifold (M,ω) equipped with a complex structure such that the corresponding almost complex structure is compatible with ω.

An **almost Kähler** manifold is a symplectic manifold equipped with a compatible almost complex structure.

Compatibility implies: if $v \neq 0$, then $\omega(v, Jv) > 0$, hence $\neq 0$.

Defn. If (M,J) is almost complex, a submanifold $N \subseteq M$ is **almost complex** if $J(TN) = TN$.

Corollary Let (M,ω,J) be almost Kähler. Then every almost complex submanifold is symplectic.

We will soon see that $\mathbb{P}^n(\mathbb{C})$ is symplectic.

Corollary All smooth projective manifolds are symplectic.

The 2-form is obtained from a projective embedding.

Recall: $\mathbb{C}^{n+1}\setminus\{0\} \xrightarrow{\pi} \mathbb{P}^n(\mathbb{C}) = \dfrac{\mathbb{C}^{n+1}\setminus\{0\}}{\mathbb{C}^\times}$

$$\cong Gr_{\mathbb{C}}(n+1,1)$$

$\exists!$ complex structure on $\mathbb{P}^n(\mathbb{C})$ such that π is holomorphic. A complex atlas is given by

$$U_j = \{[z_0 : z_1 : \cdots : z_n] \mid z_j \neq 0\}$$

for $0 \leq j \leq n$.

$$\varphi_j : U_j \longrightarrow \mathbb{C}^n$$
$$[z_0 : \cdots : z_n] \mapsto \left(\frac{z_1}{z_j}, \ldots, \widehat{\frac{z_j}{z_j}}, \ldots, \frac{z_n}{z_j}\right)$$

Symplectic point of view:

$$S^{2n+1} \xrightarrow{\pi} \frac{S^{2n+1}}{S^1} = \mathbb{P}^n(\mathbb{C})$$

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook, "i"] \arrow[d, "\pi"] & (\mathbb{C}^{n+1}, \omega_{std}) \\ (\mathbb{P}^n(\mathbb{C}), \omega_{FS}) & \end{tikzcd}$$

$\exists!$ 2-form ω_{FS} on $\mathbb{P}^n(\mathbb{C})$ such that

$$\pi^*\omega_{FS} = i^*\omega_{std}$$

so that ω_{FS} is closed and nondegenerate, called the Fubini-Study form.

Moreover, ω_{FS} is compatible with the complex structure on $\mathbb{P}^n(\mathbb{C})$.

↳ $\mathbb{C}^{n+1}\setminus\{0\} \xrightarrow{\pi} \mathbb{P}^n(\mathbb{C})$ is a principle $\mathbb{C}^\times$-bundle.

↳ $S^{2n+1} \xrightarrow{\pi} \mathbb{P}^n(\mathbb{C})$ is a principle S^1-bundle.

Reminders: Let G be a Lie group. A principal G-bundle is a manifold P together with a right G-action and a manifold B and a smooth G-invariant map $\pi: P \to B$ such that for all $b \in B$ $\exists$ nbhd $U \ni b$ and a diffeomorphism $\pi^{-1}U \cong U \times G$ which intertwines the maps to U, and which is G-equivariant where G acts on $U \times G$ by right multiplication.

Note: The G-action is free and proper. I.e. the map
$$\begin{aligned} M \times G &\to M \times M \\ (m,g) &\mapsto (m, m.g) \end{aligned}$$
is proper.

Theorem For any free, proper, right G-action $P \circlearrowleft G$, $\exists!$ manifold structure on $B := P/G$ such that the quotient map $\pi: P \to B$ is a principle G-bundle.

Remark: If G is compact, any G-action on any manifold is proper.

Given a principal G-bundle $\pi : P \to B$ and a differential form α on P $\exists$ a diff. form β on B such that $\pi^*\beta = \alpha$ if and only if:

α is G-invariant and α is horizontal:

$$\boxed{\text{For all } v \text{ tangent to the } G\text{-orbit, } v \lrcorner \alpha = 0.}$$

such α is called basic.

Fubini-Study:

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook, "i"] \arrow[d, "\pi"'] & \mathbb{C}^{n+1} \\ \mathbb{P}^n(\mathbb{C}) & \end{tikzcd}$$

We need to show that $i^* \sum_{j=0}^{n} dx_j \wedge dy_j$ is basic, i.e. S^1 invariant and S^1-horizontal.

$\hookrightarrow \sum_{j=0}^{n} dx_j \wedge dy_j$ is invariant and i is equivariant

$\Rightarrow i^* \sum dx_j \wedge dy_j$ is invariant.

$\hookrightarrow$ At each point the tangent to the S^1-orbit is ξ.

$$\xi = \sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}$$

$$\xi \lrcorner\, i^* \sum dx_j \wedge dy_j = i^* \sum_j x_j(-dx_j) - y_j(dy_j)$$

$$= -i^* \cdot \frac{1}{2} d\left(\sum_{j=0}^{n} (x_j^2 + y_j^2)\right) = -\frac{1}{2} d \underbrace{i^*(x_j^2 + y_j^2)}_{\equiv 1} = 0$$

This is a special case of symplectic reduction.