Ex 2.4 Solve one of these two problems:

- $\mathcal{J}(V,\omega) \subseteq \mathrm{End}(V)$ is closed
- The map $Sp(V,\omega) \to \mathcal{J}(V,\omega)$ is proper.

Pf ctd...

more useful

↳ π is smooth in the diffeological sense:

If g_y, $y \in V \underset{\text{open}}{\subseteq} \mathbb{R}^r$ $(r \in \mathbb{N})$ is a smooth family of Riemannian metrics, then $\pi(g_y)$ is a smooth family of compatible Riemannian metrics

↳ π is continuous in the C^∞ topology:

$g_n \longrightarrow g \iff \forall k \; g_n^{(k)} \longrightarrow g^{(k)}$ uniformly on compact subsets (contained in domains of coordinate charts).

subbasis for topology:

↳ Fix an atlas $\mathcal{A}$, chart $\mathcal{U}$, compact $K \subset \mathcal{U}$, diff. operator $D = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$, open set of matrices, ... and many other things (see the notes).

Exercise (4.1) "Flexibility of almost complex structures"

Let (M,ω) be symplectic manifold and $U \subseteq M$ open. Let J_1 be a compatible almost complex structure defined on a neighbourhood of the closure $\overline{U}$. Then there exists a compatible almost complex structure J_2 on M such that $J_2|_U = J_1|_U$.

Examples of lagrangian submanifolds

(1) Consider (M,ω) symplectic and a diffeo.

$$f: M \longrightarrow M$$

Take the graph $\Gamma_f = \{(x,f(x)) : x \in M\} \subset M \times M$ is lagrangian with respect to $(-\omega)\oplus\omega$ if and only if f is a symplectomorphism, i.e. $f^*\omega = \omega$.

Indeed: parameterize through
$$\begin{aligned} \gamma : M &\longrightarrow \Gamma_f \\ x &\longmapsto (x, f(x)) \end{aligned}$$

Then $\gamma^*((-\omega)\oplus\omega) = -\omega \oplus f^*\omega$

Remark: One can build a category with objects symplectic manifolds and morphisms given by lagrangian submanifolds of the product: $\mathrm{Hom}(M,N) := \{\mathcal{L} \subset M \times N \text{ lagrangian}\}$. Think of the "monad in the bicategory of spans" formulation of a category (and actually understand it (note to self)).

Potential project (for anyone reading this, be warned our enthusiasms may differ).

(2) Let N be any manifold and $\beta \in \Omega^1(N)$.

$\Gamma_\beta = \{(x, \beta|_x) : x \in N\}$ is lagrangian in T^*N if and only if $d\beta = 0$.

Indeed: parameterize the graph by

$$\begin{aligned} \gamma : N &\longrightarrow T^*N \\ x &\longmapsto (x, \beta|_x) \end{aligned}$$

Then $\gamma^* \alpha_{taut} = \beta$, so $d\beta = -\gamma^* \omega_{can}$.

Note: If β is exact ($\beta = dh$ for $h \in C^\infty(N)$), then

$$\operatorname{crit} h := \{x : dh|_x = 0\} \stackrel{\approx}{=} \Gamma_\beta \cap (0\text{-section})$$

where we identify N with the 0-section.

in (1), $\begin{pmatrix} \text{fixed points} \\ \text{of } f \end{pmatrix} = (\text{diagonal}) \cap \Gamma_f$

where we identify M with diag_M.

Theorem For (M, ω) symplectic, compact and simply connected, $\dim M = 2n \geq 2$, let $f : M \to M$ be a C^1-small symplectomorphism. Then f has at least two fixed points. Moreover,

$$\#\{\text{fixed points of } f\} \geq \operatorname{Crit} M := \min\{\#\operatorname{crit} h : h \in C^\infty(M)\}$$

i.e. $\exists C^1$-nbhd $\mathcal{U}$ of $\mathrm{id}_M \in C^\infty(M,M)$ s.t. $\forall f \in \mathcal{U}$
$\#\{\text{fix pt.}\} \geq \operatorname{Crit} M$.

Office hour

Defn. The compact–open topology on $\{f : M_1 \to M_2\}$ has as subbasis $\Big\{ \{f : f(K) \subseteq \mathcal{O}\} : K \subseteq M_1 \text{ compact}, \mathcal{O} \subseteq M_2 \text{ open} \Big\}$.

Defn. ~~Apply the~~ The C^k-topology is built by taking as subbasis the union of the subbases of the compact open topology on $df : TM_1 \to TM_2$ (viewed with the subspace topology) and $d^2 f : T^2 M_1 \to T^2 M_2$ and $\cdots$ and $d^k f : T^k M_1 \to T^k M_2$.

Exercise: $\mathrm{Diffeo}(M_1, M_2) \subset C^\infty(M_1, M_2)$ is not C^0-open, but is C^1-open, assuming M_1 and M_2 are compact.

The map $\pi : Sp(V, \omega) \longrightarrow Sp(V,\omega)/U(V)$ is proper: $\operatorname{cod} \pi \cong \{\text{sym. pos. def.}\} \cap Sp(V,\omega)$, so $\pi^{-1}(K) = K_1 \times U(V)$ for some K_1 compact.

2.4

↳ For sufficiently nice spaces, proper $\Leftrightarrow$ closed and with compact fibres.

↳ proper continuous bijections are homeomorphisms (proper $\Rightarrow$ closed).

↳ Compact $\to$ Hausdorff is always proper. I.e. this is a special case.

↳ ~~$GL(V) \longrightarrow$~~

$$\varphi : GL(V) \times V \longrightarrow \mathbb{R}$$
$$(A, v) \longmapsto \omega(v, Av)$$

$$f_v : GL(V) \to GL(V) \times V, \quad A \mapsto (A, v)$$

$$\bigcap_{v \in V} f_v^{-1} \varphi^{-1}(\mathbb{R}_{\geq 0}) \cap \bigcap_{u,v \in V} \big(A \mapsto \omega(u, Av) - \omega(v, Au)\big)^{-1}(0) \cap (\text{nondegenerate})$$