Remark: The same construction works for building adapted coordinates on TN. Just replace $dq_i|_x$ with $\frac{\partial}{\partial q_i}\Big|_x$, where $\left\{\frac{\partial}{\partial q_i}\Big|_x\right\}$ is the dual basis of $\{dq_i|_x\}$

On the chart $U \subset N$, $TN|_U$:

$$q_j(x,v) = q_j(x)$$
$$v_j(x,v) \text{ given by } \quad v = \sum_{j=1}^{n} v_j \frac{\partial}{\partial q_j}\Big|_x$$

Observe that $v_j = dq_j : TN \to \mathbb{R}$. Woah.

Remark Re 2.4: An intermediate step to properness!

Prove that $\mathcal{J}(V,\omega)$ is closed in $\mathrm{End}_{\mathbb{R}}(V)$.

Recall: We built ω_{can} on T^*L: $\omega_{can} = -d\alpha_{taut}$.

In adapted coordinates: $\alpha_{taut} = \sum p_j dq_j$

$$\omega_{can} = \sum dq_j \wedge dp_j$$

We are now in a position to state Weinstein's theorem:

Thm (Weinstein's Tubular Neighbourhood)

Let L be a ~~symplectic~~ Lagrangian submanifold of a symplectic manifold (M,ω). Then there exists a symplectomorphism

$$\text{neighbourhood of } L \text{ in } (M,\omega) \xrightarrow{\sim} \text{neighbourhood of } L \text{ in } (T^*L, \omega_{can})$$

Lemma For L a lagrangian submanifold of M, there exists a lagrangian splitting $TM|_L = TL \oplus E$

I.e., there exists a (fibrewise) lagrangian subbundle $E \subset TM|_L$ complementary to TL.

we don't deal with closedness

w.r.t. ω_M.

Sketch of proof. The Crucial lemma (week 2) takes an arbitrary Riemannian metric to a compatible metric, hence a compatible almost complex structure $J : TM \to TM$

Take $E = \mathrm{im}(J|_{TL}) = J(TL)$

Good Exercise:

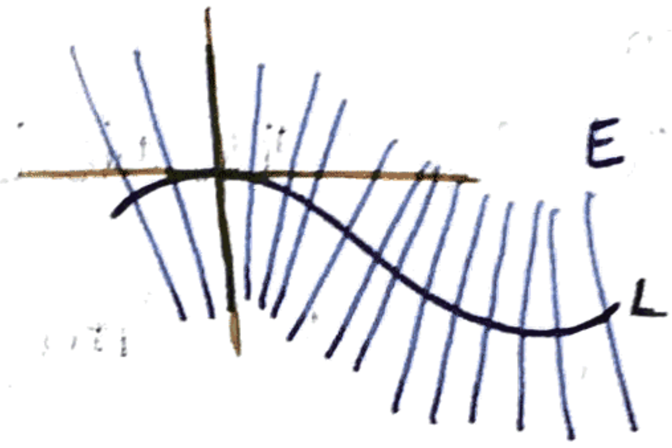
- E is lagrangian
- E is complementary to TL (enough to show they intersect trivially since the dimensions are complementary).

in fact, $\forall u,v,\ \omega(Ju, Jv) = \omega(u,v)$.

Special case: $L \xhookrightarrow[\text{zero section}]{} T^*L$

For any vector bundle $E \to L$, we have a natural splitting:

$$\underset{\uparrow \text{zero section of } TE}{TE|_L} = TL \oplus E$$



In particular,

$$\boxed{T(T^*L)\big|_L = \underset{\underset{q_i\text{'s}}{\uparrow}}{TL} \oplus \underset{\underset{p_i\text{'s}}{\uparrow}}{T^*L}}$$

Moreover, at each $x \in L =$ 0-section,

$$\omega_{can}|_x((v_1,\varphi_1),(v_2,\varphi_2)) = \varphi_2(v_1) - \varphi_1(v_2)$$

where $(v_j,\varphi_j) \in T_xL \oplus T_x^*L$. In general, $V \oplus V^*$ has a natural symplectic structure via the pairing.

More generally,

If $TM|_L = TL \oplus E$ is a lagrangian splitting, ω gives a nondegenerate pairing between TL and E, hence an isomorphism $E \cong TL$

$$\begin{tikzcd} TM|_L \arrow[r, "\omega^\#"] & T^*M|_L \arrow[r, "\iota^*"] & T^*L \\ E \arrow[u, hook] \arrow[urr, "\sim"] & & \end{tikzcd}$$

$$\ker(\omega^\# \,/\!/\, \iota^*) = TL$$

Defn $\boxed{f /\!/ g := g \circ f}$

With this identification, $TM|_L \cong TL \oplus T^*L$ and $\omega|_{TM|_L}$ becomes

$$(v_1, \varphi_1) \otimes (v_2, \varphi_2) \longmapsto \varphi_2(v_1) - \varphi_1(v_2)$$

Let $\psi : \left\{ \begin{matrix} \text{nbhd of} \\ L \text{ in } M \end{matrix} \right\} \xrightarrow[\text{diffeo.}]{\sim} \left\{ \begin{matrix} \text{nbhd of } L \\ \text{(0-sec) in } \nu L \end{matrix} \right\}$ be the diffeomorphism from the ordinary tubular neighbourhood theorem. $\psi|_L = $ 0-section $L \to T^*L$

This gives

$$d\psi|_L : TM|_L \xrightarrow{\sim} T(\nu L)|_{L = 0\text{-sec.}} = TL \oplus \nu L$$

Thm (tubular neighbourhood) v.2 For all splittings $TM|_L = TL \oplus \nu L$, $\exists \psi$ which induces this splitting.

The normal bundle is defined via

$$0 \to TL \to TM|_L \to \nu L \to 0$$

choice of splitting

(End of) proof of W.'s lagr. tub. nbhd. thm:

Given $L \subseteq M$ lagrangian submanifold, choose a lagrangian splitting $TM|_L = TL \oplus E$ (e.g. $\underset{\cong TL \oplus T^*L}{}$
$E = J(TL)$ for a compatible almost complex structure $J \in \mathrm{End}(TM)$, which is smooth by the crucial lemma)

i.e. $\forall x \in L$, $T_xM \cong T_xL \oplus E_x$.

The ordinary tubular neighbourhood theorem gives us an inducing diffeomorphism

$$\begin{array}{c} \text{nbhd of} \\ L \text{ in } M \end{array} \xrightarrow[\text{diffeo}]{\Psi \sim} \begin{array}{c} \text{nbhd of } L \\ \text{in } \mathcal{V}L \cong T^*L \end{array}$$

{last week}

Such that

$$d\Psi|_L : TM|_L \longrightarrow T(\mathcal{V}L)|_L$$
$$TM|_L = TL \oplus T^*L, \quad T(\mathcal{V}L)|_L = TL \oplus \mathcal{V}L = TL \oplus T^*M$$

is the identity map on $TM \oplus T^*M$

Recall. $\mathcal{V}L \cong T^*L$ naturally: $\forall x \in L \overset{\iota}{\hookrightarrow} M$

$$T_xM \xrightarrow{\omega^\#} T_x^*M \xrightarrow{\iota^*} T_x^*L$$
$$u \longmapsto \omega(u,\cdot) \longmapsto \omega(u,\cdot)|_L$$

$$\mathrm{Ker}(\omega^\# \,//\, \iota^*) \underset{\substack{\text{by def. of} \\ \text{orthocomplement}}}{=} L^\omega \underset{\substack{\text{by def. of} \\ \text{"lagrangian"}}}{=} L$$

Note: If $V = S \oplus E$ is a lagrangian splitting, then ω induces $E \cong S^*$. If $u \in S$ and $\varphi \in S^* = E$,

$$\omega(u, \varphi) = \varphi(u)$$

i.e. $\omega((u,0),(0,\varphi)) = \varphi(u)$

By antisymmetry, this yields:

$$\omega((u_1, \varphi_1), (u_2, \varphi_2)) = \varphi_2(u_1) - \varphi_1(u_2)$$

So that's where this formula comes from!

Remark: Symplectic geometry over Banach spaces requires that $\omega^\#$ is a linear isomorphism; this is called strong nondegeneracy.

...So Ψ satisfies $\forall x \in L$

$$\Psi^* \omega_{can}\big|_x = \omega\big|_x$$

(since $\omega_{can}|_x$ and $\omega|_x$ are both the standard symplectic form on $T_xL \oplus T_x^*L$ and $d\Psi|_x$ respects the isomorphisms with $T_xL \oplus T_x^*L$).

Pf... By Weinstein's local normal form,

$$\exists \tilde{\Psi} : \begin{matrix}\text{neighbourhood}\\ \text{of } L \text{ in } M\end{matrix} \longrightarrow \begin{matrix}\text{neighbourhood}\\ \text{of } L \text{ in } M\end{matrix}$$

such that $\tilde{\Psi}^*(\psi^*\omega_{can}) = \omega$ and $\tilde{\Psi}|_L = \mathrm{Id}_L$

So $\psi \circ \tilde{\Psi} : \begin{matrix}\text{neighbourhood}\\ \text{of } L \text{ in } M\end{matrix} \longrightarrow \begin{matrix}\text{neighbourhood}\\ \text{of } L \text{ in } T^*L\end{matrix}$

satisfies $(\psi \circ \tilde{\Psi})^*\omega_{can} = \omega$. ∎

Compatible Almost Complex Structures

Defn. An almost complex structure on a manifold M is a smooth bundle automorphism $J \in \mathrm{Aut}(TM)$ such that $J^{\circ 2} = -\mathrm{Id}_{TM}$.

If $M = (M, \omega)$ is symplectic, J is called compatible if $\forall x \in M$, J_x is compatible with ω_x on (T_xM, ω_x).

Defn. $\mathcal{J}(M,\omega) := \left\{ \begin{matrix}\text{compatible almost}\\ \text{complex structures}\end{matrix} \right\}$

$$= \left\{ C^\infty \text{ sections of } \mathrm{Aut}(TM) \text{ whose fibre at } x \in M \text{ is } \mathcal{J}(T_xM, \omega|_x) \right\}$$

Lemma Let (M,ω) be a symplectic manifold. Then $\mathcal{J}(M,\omega)$ is nonempty. Moreover, it is contractible.

<u>Remark</u>: $\mathcal{J}(M,\omega)$ is a Fréchet manifold with the C^∞ topology.

<u>Proof</u>. The <u>crucial lemma</u> gives for each $x \in M$ a retraction

$$\pi_x : \left\{ \begin{array}{c} \text{inner products} \\ \text{on } T_x M \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{compatible inner} \\ \text{products on } T_x M \end{array} \right\}$$

These fit into a retraction (is it smooth? continuous? what does that mean?)

$$\pi : \left\{ \begin{array}{c} \text{Riemannian metrics} \\ \text{on } M \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{compatible R.} \\ \text{metrics on } M \end{array} \right\} \cong \mathcal{J}(M,\omega)$$

Fix any metric g_0

↳ $\pi(g_0)$ give a compatible metric, so $\mathcal{J}(M,\omega) \neq \emptyset$

↳ $g \mapsto \pi((1-t)g + tg_0)$ gives a homotopy between the identity map and a constant map on $\left\{ \begin{array}{c} \text{compatible} \\ \text{R. metrics} \end{array} \right\} \cong \mathcal{J}(M,\omega)$.