Lagrangian Submanifolds & Cotangent Bundles

<u>Recall</u>: A linear subspace $S \subset (V, \omega)$ of a symplectic vector space, $\dim V = 2n$, is

- ↳ symplectic if $\omega|_S : S \times S \to \mathbb{R}$ is nondegenerate
- ↳ isotropic if $\omega|_S = 0$; i.e. $S \subseteq S^\omega$
- ↳ <u>coisotropic</u> if $S \supseteq S^\omega$
- ↳ <u>lagrangian</u> if $S = S^\omega \Leftrightarrow S$ isotropic, $\dim S = n$

↳ S and S^ω have <u>complementary</u> dimensions: $\iota : S \hookrightarrow V$ inclusion, then

$$\begin{tikzcd} V \arrow[r, "\omega^\#"] \arrow[rr, bend right, "\varphi"'] & V^* \arrow[r, "\iota^*"] & S^* \end{tikzcd}$$

$\operatorname{Ker}\varphi = S^\omega$, $\operatorname{rank}\varphi = \operatorname{rank}\iota^* = \operatorname{rank}\iota = \dim S$

By rank-nullity, $\operatorname{codim} S^\omega = \dim S$.

Let now $i: N \longrightarrow (M^{2n}, \omega)$ be an embedding (or weak embedding / immersion). Then

Defn. N is:

- **symplectic** if $TN \subseteq TM|_N$ is fibrewise symplectic
  i.e. $i^*\omega$ is symplectic on N.
- **isotropic** ——— " ——— isotropic
  i.e. $i^*\omega = 0$
- **coisotropic** ——— " ——— coisotropic
- **lagrangian** ——— " ——— lagrangian
  i.e. $i^*\omega = 0$ and $\dim N = n$.

Consider $S \subset (V, \omega)$ a lagrangian subspace of V, $\dim V = 2n$

$$\begin{tikzcd} V \arrow[r, "\omega^\#|_S"] \arrow[d] & S^* \\ V/S \arrow[ur, dashed, "\sim"] & \end{tikzcd}$$

$S = S^\omega = \ker \omega^\#|_S$

Thus we get, with a minor ~~abuse~~ expansion of notation,

$$\omega^\#|_S : V/S \xrightarrow{\sim} S^*$$

This map is natural in the following senses:

↳ It is $Sp(V)^S$-equivariant

↳ It is "smooth in S". Allowing S to vary across lagrangian ~~submanifolds~~ subspaces. Denote by **$LaGr(V,\omega)$** $:= \{S \subset (V,\omega) \text{ lagrangian}\}$ $LaGr(V,\omega) \subseteq Gr(V,n)$. Consider vector bundles Q and D so that $Q|_S = V/S$, $D|_S = S^*$. We then get

$$\begin{tikzcd} Q \arrow[rr, "\omega^{\#}|_S"] \arrow[dr] & & D \arrow[dl] \\ & LaGr(V,\omega) & \end{tikzcd}$$

where $\omega^{\#}|_S$ is now an extension to a smooth vector bundle isomorphism.

For $i: L \hookrightarrow (M,\omega)$ a lagrangian submanifold, we get

$$\underbrace{TM|_L / TL}_{\substack{\nu L \\ \text{the normal} \\ \text{bundle of } L \text{ in } M}} \xrightarrow{\sim} \underbrace{T^*L}_{\substack{\text{cotangent} \\ \text{bundle of} \\ L}}$$

(ordinary) Tubular neighbourhood thm

Let $L \subseteq M$ be embedded. Then we can find:

a neighbourhood of L in M $\underset{\text{diffeo.}}{\cong}$ a neighbourhood of 0-section of $\mathcal{V}L$

whose restriction to L is the 0-section.

Thm (Alan Weinstein) We can do this symplectically with respect to a natural symplectic structure on T^*L

Cotangent Bundles

For N a manifold, $M := T^*N$ has a manifold structure.

$$M = \coprod_{x \in N} T_x^*N \text{ as a set}$$

$$T_x^*N := (T_xN)^*$$

$(q,v) \in TN$, $q \in N$, $v \in T_q^*N$

If $q_1, \dots, q_n$ are coordinates on N, then

$q_1, \dots, q_n, v_1, \dots, v_n$ are adapted coordinates

on TN, where $q_i(x,v) := q_i(x)$

$\rightsquigarrow q_1, \ldots, q_n, p_1, \ldots, p_n$ coordinates on T^*N

From this we build local trivializations; this gives us M's manifold structure.

So $\{dq_i|_x\}_{i=1}^n$ is a basis of T^*_xN

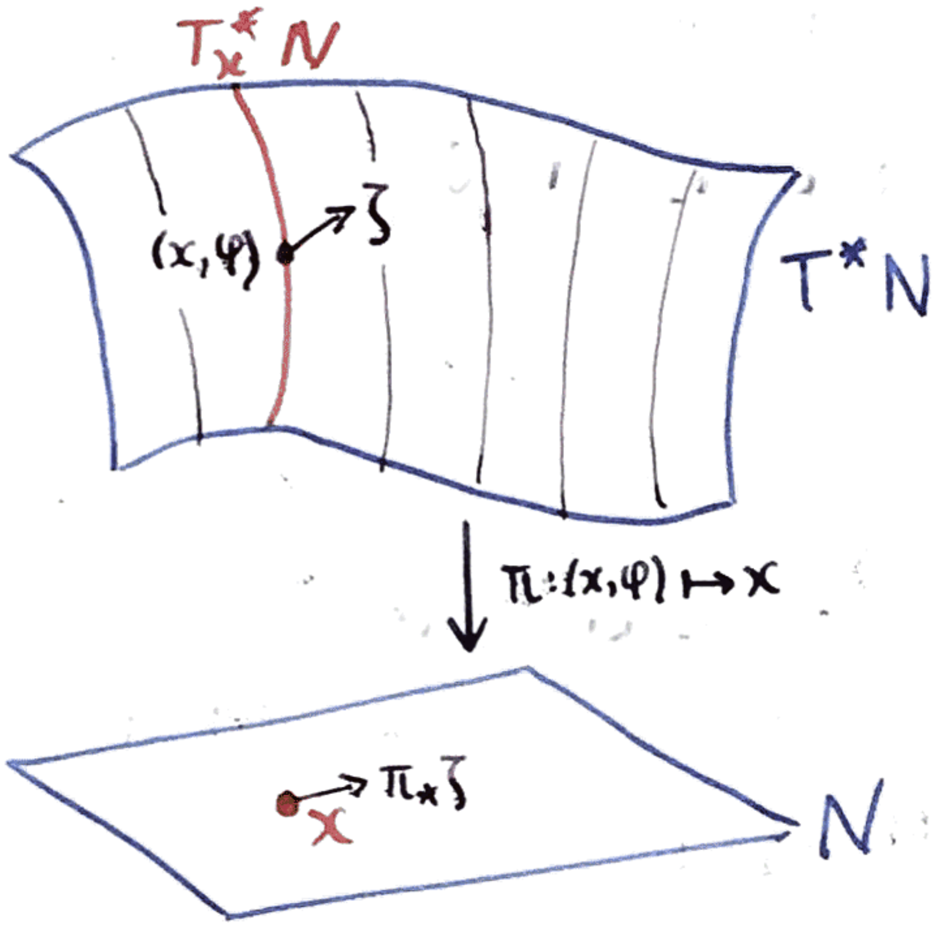
Define p_i so that for $\varphi \in T^*_xN$,

$$\varphi = \sum_{j=1}^n p_j dq_j|_x$$

$p_j = p_j(x, \varphi)$

These are declared smooth by definition.

Defn. The *tautological 1-form* α on T^*N



For $\zeta \in T_{(x,\varphi)}(T^*N)$, $\pi_*\zeta \in T_xN$

$$\alpha(\zeta) := \varphi(\pi_*\zeta)$$

In adapted coordinates,

$$\pi(q_1, \ldots, q_n, p_1, \ldots, p_n) = (q_1, \ldots, q_n)$$

$$\zeta = \sum_j a_j \frac{\partial}{\partial q_j} + b_j \frac{\partial}{\partial p_j}, \quad \pi_*\zeta = \sum_j a_j \frac{\partial}{\partial q_j}$$

$$\varphi = \sum_{j=1}^n p_j dq_j, \quad \alpha(\zeta) = \sum a_j p_j = \sum p_j (dq_j(\zeta))$$

We want to say $\alpha = \sum_{j=1}^n p_j dq_j$. Set now

$$\omega_{can} := -d\alpha = \sum dq_j \wedge dp_j$$

Thus we get a canonical symplectic form ω_{can} on T^*N; any adapted coordinates are a Darboux chart.

Remark: The same construction works for building adapted coordinates on TN. Just replace $dq_i|_x$ with $\frac{\partial}{\partial q_i}\Big|_x$, where $\left\{\frac{\partial}{\partial q_i}\Big|_x\right\}$ is the dual basis of $\{dq_i|_x\}$

On the chart $U \subset N$, $TN|_U$:

$$q_j(x,v) = q_j(x)$$
$$v_j(x,v) \text{ given by } v = \sum_{j=1}^{n} v_j \frac{\partial}{\partial q_j}\Big|_x$$

Observe that $v_j = dq_j : TN \to \mathbb{R}$. Woah.