↳ Ana Da Silva: good book

→ The Darboux theorem (see lecture 1) relies on the closedness of ω. In the vector space case, $T_v V \cong V$ naturally, so closedness is there also.

→ Older proofs were done in coordinates; see older texts for such an approach, e.g. [Shlomo Sternberg]: "Lectures on Differential Geometry" (1964)

→ We will use Weinstein's proof (1969), which ~~will~~ uses Moser's Method (~1965)

↳ This technique works for Banach manifolds

Thm (Darboux) Let (M,ω) be a symplectic manifold of dimension $2n$. Recall ω is a closed, nondegenerate 2-form. Then $\forall m \in M$ there is a diffeomorphism

$$\underset{\substack{\cap \text{ open nbhd of } m \\ (M,\omega)}}{\varphi : \mathcal{U}} \xrightarrow{\sim} \underset{\substack{\cap \text{ open} \\ (\mathbb{R}^{2n}, \omega_{std})}}{\Omega}$$

such that $\varphi^* \omega_{std} = \omega$. That is, writing $\varphi = (x_1, y_1, x_2, y_2, \text{—}, x_n, y_n)$, we have

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

More generally, we have:

Theorem (local normal form)

For M a manifold and $N \subseteq M$ an embedded submanifold, then given closed 2-forms ω_0, ω_1 defined on neighbourhoods of N in M such that:

- ↳ ω_0 and ω_1 agree on N, and
- ↳ ——— " — are nondegenerate on N.

(i.e. $\forall m \in N$, $\omega_0|_m = \omega_1|_m$ are nondegenerate as symplectic tensors on $T_m M$),

there exist neighbourhoods $\mathcal{U}_0, \mathcal{U}_1$ of N in M ~~such that~~ and a diffeomorphism $\psi : \mathcal{U}_0 \longrightarrow \mathcal{U}_1$ which fixes N pointwise such that $\psi^* \omega_1 = \omega_0$

Remark: Given a diffeomorphism φ, we can define pushforwards (of vector fields, diff. forms, etc.) of $\varphi_* := (\varphi^{-1})^*$

Remark: This theorem gives us a pointwise $\Rightarrow$ local result.

Proof (LNF). Let's interpolate:

$$\omega_t := (1-t)\omega_0 + t\omega_1 \text{ for } 0 \leq t \leq 1$$

$\hookrightarrow$ ω_t is a smooth family of closed, 2-forms defined near N

$\hookrightarrow$ On $TM|_N$, we know $\omega_0 = \omega_1 = \omega_t$

It follows ω_t is nondegenerate on $TM|_N$; as this is an open condition, ω_t is nondegen. near N!

So $\alpha_{t_0} := \frac{d}{dt}\omega_t\Big|_{t=t_0} = \lim_{t \to t_0} \frac{\omega_t - \omega_{t_0}}{t - t_0}$ is a smooth family of closed 2-forms near N which vanish along N (i.e. on $TM|_N$).

We use the following Lemma:

Lemma (Relative Poincaré) Let $i: N \hookrightarrow M$ be an embedded submanifold and α_t a smooth family of closed k-forms near N whose pullback to N is zero (i.e. $i^*\alpha_t = 0$ (i.e. α_t vanishes on TN)). Then there exists a smooth family of $(k-1)$-forms β_t near N such that $d\beta_t = \alpha_t$ near N. Moreover, if α_t vanishes along N (see above $\uparrow$), then the β_t can be chosen so that they also vanish along N!

Moser's Trick

Seek out a time-dependant flow $\psi_t : \begin{matrix}\text{nbhds}\\ \text{of } N\end{matrix} \longrightarrow \begin{matrix}\text{nbhds}\\ \text{of } N\end{matrix}$.
(i.e. $(t,m) \mapsto \psi_t(m)$ is a smooth map from an open neighbourhood W of $[0,1]\times N$ in $[0,1]\times M$ and for all t, ψ_t is a diffeo. from $W_t := \{m\in M \mid (t,m)\in W\} = (\{t\}\times M)\cap W$ to some neighbourhood of N in M.) So that $\psi_0 = \mathrm{Id}$ and for all t, $\psi_t^* \omega_t = \omega_0$

- Time-dependant means we do not necessarily have an $\mathbb{R}$-action ($\psi_{t_1}\circ\psi_{t_2} \neq \psi_{t_1+t_2}$)
- The velocity v.f. will be a time-dependant v.f. X_t defined near N such that

$$\boxed{\frac{d}{dt}\psi_t = X_t \circ \psi_t}$$

$\forall m \in M$, we get $\gamma : [0,1] \longrightarrow M$
$t \longmapsto \psi_t(m)$

Recall from ODE's:

Given a time-dependant v.f. X_t, $0\leq t\leq 1$ defined near N such that $\forall t$ $X_t|_N = 0$, $\exists!$ time-dependant flow $\psi_t : \begin{matrix}\text{nbhds}\\ \text{of } N\end{matrix} \longrightarrow \begin{matrix}\text{nbhds}\\ \text{of } N\end{matrix}$ such that $\psi_0 = \mathrm{Id}$ and

$$\frac{d\psi_t}{dt} = X_t \circ \psi_t .$$

So, it will be sufficient to find X_t such that the corresponding ψ_t satisfies $\boxed{\frac{d}{dt}\psi_t^*\omega_t = 0}$

Let us differentiate:

Leibniz-like rule for time-dependant flows

$X_t \circ \psi_t = \frac{d}{dt}\psi_t$

$$\frac{d}{dt}\psi_t^*\omega_t \overset{\downarrow}{=} \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right)$$

(explaination to follow)

$$\overset{\text{want}}{=} 0$$

It is enough to require: $\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = 0$

It's time for Cartan's Magic Formula!

$$\mathcal{L}_{X_t}\omega_t \overset{\downarrow}{=} d\iota_{X_t}\omega_t + \iota_{X_t}\cancel{d\omega_t}^{\,0}$$

$$= d\iota_{X_t}\omega_t$$

Define $\alpha_t := \frac{d}{dt}\omega_t$. By Relative Poincaré (still not yet proven ☹), $\exists$ smooth family of 1-forms β_t near N such that $d\beta_t = \alpha_t$ for all t, and such that β_t vanish along N (i.e. on $TM|_N$).

So $\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = d(\beta_t + \iota_{X_t}\omega_t)$

Thus it is enough to find X_t such that

$$\underbrace{\beta_t}_{\text{given}} + \underbrace{\iota_{X_t}}_{\text{seek}}\omega_t \overset{\text{want}}{=} 0$$

A pleasant closed 1-form

I.e. $\beta_t = -\iota_{X_t}\omega_t$. For all points m near N,

$$\omega_t^{\#} : T_m M \xrightarrow{\sim} T_m^* M$$
$$x \longmapsto \iota_x\omega_t|_m$$

So $X_t := (\omega_t^{\#})^{-1}(-\beta_t)$. Because $\beta_t = 0$ along N, $X_t = 0$ along N. "Integrate" (solve the ODE)

$\rightsquigarrow \psi_t : \text{nbhd of } N \to \text{nbhd of } N$ s.t. $\psi_t|_N = \mathrm{Id}$, $\psi_t^* \omega_t = \omega_0$

Note: Since $\omega_t = (1-t)\omega_0 + t\omega_1$, $\alpha_t = \frac{d}{dt}\omega_t = \omega_1 - \omega_0$ is <u>time-independant</u>! So in the Relative Poincaré lemma, the parameter t was unneeded.

↳ this will also be the case in the exercise

Exercise Let M be a closed (compact, $\partial M = \phi$) 2-manifold. Let ω_0, ω_1 be area forms (i.e. nonvanishing 2-forms on M. Suppose

$$\int_M \omega_0 = \int_M \omega_1 > 0$$

and that ω_0, ω_1 induce the same orientation on M. Then $\exists$ a diffeomorphism $\psi : M \to M$ such that $\psi^* \omega_1 = \omega_0$. That is, population density maps of earth are diffeomorphic to the original globe ☺.

↳ Moser's 1965 paper is a good resource when solving this exercise.

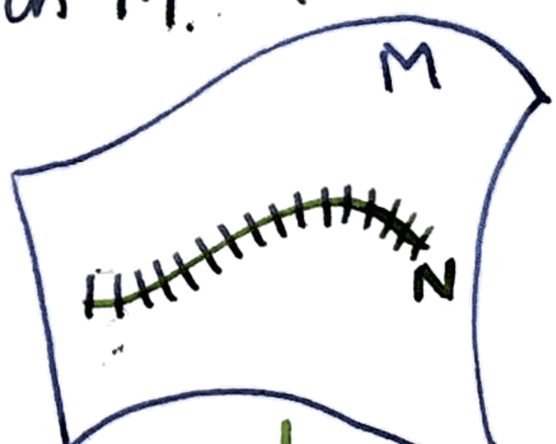
Homotopy Property Let V be a manifold. Define $i_s : V \to [0,1]\times V$ by $i_s(x) = (s,x)$. Then $\forall \gamma \in \Omega^k([0,1]\times V)$,

$$i_1^*\gamma - i_0^*\gamma = (\pi_V)_* d\gamma + d(\pi_V)_*\gamma$$

Where $\pi_V : [0,1]\times V \to V$ is projection to V and $\forall k$ $(\pi_V)_* : \Omega^k([0,1],V) \to \Omega^{k-1}(V)$ is the pushforward given by fibre integration. I.e. $(\pi_V)_*$ is a homotopy operator between i_0^* and i_1^* as morphisms of differential complexes

$$i_0^*, i_1^* : (\Omega^\bullet([0,1]\times V), d) \to (\Omega^\bullet(V), d)$$

Corollary Let $N \subseteq M$ be a submanifold. Let V be a tubular neighbourhood of N in M. I.e., V is a neighbourhood of N in M together with a diffeomorphism $V \cong \nu N := TM|_N/_{TN}$ (the normal bundle of N in M) [This is diffeomorphic to the (open) disc bundle of codim dim$N$], whose restriction to N is the zero section.



Define $R_t : V \to V$ fibrewise multiplication ~~of~~ by $t \in [0,1]$. $R : [0,1]\times V \to V$, $R(t,v) = R_t(v) = t\cdot v$. $R\circ i_1 = \mathrm{Id}_V$, $\pi : R\circ i_0 : V \to N$ projection

By the homotopy formula for $\gamma = R^*\alpha$ $(\alpha \in \Omega^k(V), \gamma \in \Omega^k([0,1]\times V))$

$$\alpha - R^*\alpha = (\pi_V)_* dR^*\alpha + d(\pi_V)_* R^*\alpha$$

↳ If α is closed, then $dR^*\alpha = R^*d\alpha = 0$

↳ If its pullback to N vanishes, then $R^*\alpha = 0$

We get $\alpha = d((\pi_V)_* R^*\alpha)$. For a smooth family α_t of closed k-forms on V that vanish on TN, take $\beta_t := (\pi_V)_* R^*\alpha_t$. Also, if α_t vanishes on $TM|_N$, then so does β_t.