This gives us

$$\left\{\begin{matrix}\text{symmetric}\\ \text{pos. definite}\\ \text{mat. in } Sp(\mathbb{R}^k)\end{matrix}\right\} \xrightarrow[\text{diffeo.}]{\sim} Sp(\mathbb{R}^k)/U(n)$$

But also:

$$Sp(\mathbb{R}^{2n})/U(n) \xrightarrow[\text{diffeo.}]{\sim} \mathcal{J}(V,\omega)$$

$$AU(n) \longmapsto A_* \underbrace{J_0}_{\substack{\text{standard complex}\\ \text{structure on } \mathbb{R}^{2n}}}$$

This map is onto by Gram-Schmidt for Hermitian structures.

It still remains to show:

$$\left\{\begin{matrix}\text{symmetric}\\ \text{pos. def.}\\ \text{mat in } Sp(\mathbb{R}^{2n})\end{matrix}\right\} \cong \mathbb{R}^m \quad \text{for some } m.$$

Defn. A **Hermitian structure** on a real vector space V is a choice of compatible

- inner product g
- symplectic tensor ω
- complex structure J

$$\forall u,v \in V \quad g(u,v) = \omega(u, Jv)$$

This is equivalent to $H = g + i\omega : V \times V \to \mathbb{C}$ being a Hermitian inner product.

Recall: This implies the existence of a basis $\{v_1, \ldots, v_n\} \subset V$ over $\mathbb{C}$ s.t. $H(v_i, v_j) = \delta_{ij}$.

This is equivalent to $(V, J, H) \cong (\mathbb{C}^n = \mathbb{R}^{2n}, i, (z,w) \mapsto \sum_i \bar{z}_i w_i)$

Proof. Hermitian Gram-Schmidt

Recall: A **Lie group** is a group object in the category C^∞-mfld. I.e. G together with $\mu: G \times G \to G$, $(a,b) \mapsto ab$, $i: G \to G$, $a \mapsto a^{-1}$ both of which are smooth.

$$\boxed{\textbf{Fact} \quad U(n),\ Sp(\mathbb{R}^{2n}) \text{ are Lie groups as embedded submanifolds of } \mathrm{End}(\mathbb{R}^{2n})}$$

Proof. See, e.g. [John Lee].

Exercise: Consider the map
$$Sp(V,\omega) \to \mathcal{J}(V,\omega)$$
$$A \mapsto A_* J_0 = A \circ J_0 \circ A^{-1}$$

ⓐ It induces a bijection
$$Sp(V,\omega)/U(V) \longrightarrow \mathcal{J}(V,\omega)$$
(Hint: Use Gram-Schmidt)

ⓑ The map is proper.

Fact If G is a Lie group and $H \subset G$ a closed subgroup, then G/H is a manifold (i.e. $\exists!$ manifold structure such that $G \to G/H$ is a submersion. Indeed, G is a principle H-bundle.

See, eg, appendix of Ginzburg-Guillemin-Karshon

Fact Let G be a Lie group and M a mfld. Suppose $G \circlearrowright M$ and $x \in M$. Let $H = \mathrm{Stab}_G\{x\} = \{g \in G \mid g.x = x\}$. The map

$$\begin{aligned} G &\longrightarrow M \\ a &\longmapsto a.x \end{aligned}$$

induces an injective immersion $G/H \longrightarrow M$. Moreover, this map is a weak embedding (or diffeological embedding).

Note: embedding $\subsetneq$ weak embedding $\subsetneq$ injective immersion

$t \mapsto [t, \alpha t]$ for $\alpha \notin \mathbb{Q}$

is a diffeological embedding, but not an embedding.

$(0,1) \longrightarrow$ 

injective immersion which is not a weak embedding.

We leave "weak embedding" undefined, but know this:

If $i: X \longrightarrow M$ is a weak embedding, $\exists !$ mfld structure on X such that i is an immersion.

$$Sp(V,\omega) \curvearrowright End_{\mathbb{R}}(V) \ni J_0$$

By the exercise and the facts on group actions, we get $Sp(V,\omega)/U(n) \longrightarrow \mathcal{J}(V,\omega)$ is a proper weak embedding $\Rightarrow$ embedding.

In Summary: This is what it means for $\mathcal{J}(V,\omega)$ to "be" a manifold. So:

$$\mathcal{J}(\mathbb{R}^{2n},\omega) \xleftarrow[\text{diffeo.}]{\sim} Sp(\mathbb{R}^{2n})/U(n)$$

$$\cong$$

$$\left\{\begin{array}{l}\text{symmetric}\\\text{matrices}\\\text{in } \mathfrak{sp}(\mathbb{R}^{2n})\end{array}\right\} \underset{\underbrace{\log}_{\substack{\text{Details on}\\\text{course website}\\\text{to follow}}}}{\overset{\exp}{\rightleftarrows}} \left\{\begin{array}{l}\text{symmetric,}\\\text{pos. definite}\\\text{mat. in } Sp(\mathbb{R}^{2n})\end{array}\right\}$$

Let's do the crucial lemma again:

Crucial Lemma There exists an $Sp(V,\omega)$-equivariant smooth strong deformation retraction from

$$\left\{\begin{matrix}\text{inner products}\\ \text{on } V\end{matrix}\right\} \longrightarrow \mathcal{G}(V,\omega)$$

i.e. $\pi_t : \left\{\begin{matrix}\text{inner products}\\ \text{on } V\end{matrix}\right\} \circlearrowleft$ such that

- $(t,g) \to \pi_t g$ is smooth
- π_0 is the identity map
- $\pi_t g = g \quad \forall g \in \mathcal{G}(V,\omega) \ \forall t \in [0,1]$
- $\operatorname{im} \pi_1 = \mathcal{G}(V,\omega)$

Proof. See Yael's notes.

Office Hours

- There are applications of symplectic geometry to Low dimensional topology (Haegaard-Floer homology)
- There are aspects of symplectic geometry which are unproven but widely accepted; gaps are being filled in over time.
- $\dim W^{\Omega} = \operatorname{codim} W$ for $W \subset V$ symplectic

Fact $\omega^{\#} : V \xrightarrow{\sim} V^*$

- Can you show this?
- $S \subseteq V \Rightarrow \operatorname{Ann}(S) \subseteq V^*$
$\dim S + \dim \operatorname{Ann}(S) = \dim V$