Legendre Transform

$\mathcal{L}: TQ \to \mathbb{R}$ lagrangian,

$\Psi: TQ \to T^*Q$ where $p = \left.\frac{\partial \mathcal{L}}{\partial v}\right|_{(x,v)}$

$(x,v) \mapsto (x,p)$

Assume Ψ is invertible (e.g. true if $\mathcal{L}$ is a fibrewise positive-definite quadratic form).

$H: T^*Q \to \mathbb{R}$ hamiltonian

$$H(x,p) = \langle p, v\rangle - \mathcal{L}(x,v) \quad \text{for} \quad v = \Psi^{-1}(x,p).$$

Thm A curve $(x(t),p(t))$ in T^*Q satisfies hamilton's equations for H if and only if

i) The corresponding curve in TQ is a <u>tangent lift</u> (= <u>prolongation</u>) of a curve in Q (necessarily $x(t)$). That is,

$$(x(t),p(t)) = \Psi(x(t), \dot{x}(t))$$

ii) <u>and</u>, $x(t)$ satisfies the Euler-Lagrange equations for $\mathcal{L}$.

Proof. Let (x,v) be adapted coordinates on TQ and (x,p) ——— " ——— T^*Q.

$\omega = \sum_j dx_j \wedge dp_j$, Hamilton's equations: $\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$

$H(x,p) = \langle p, v\rangle - \mathcal{L}(x,v)$, $v = \Psi^{-1}(x,p)$

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\langle p, v\rangle - \frac{\partial}{\partial p}\mathcal{L}(x,v) = \underbrace{v + \left\langle p, \frac{\partial v}{\partial p}\right\rangle}_{\text{Leibniz}} - \underbrace{\left\langle \overset{p}{\frac{\partial \mathcal{L}}{\partial v}}, \frac{\partial v}{\partial p}\right\rangle}_{\text{chain rule}}$$

$= v$

So hamilton's first equation is $\dot{x} = v$, which is condition (i).

Pf... $-\frac{\partial H}{\partial x} = -\frac{\partial}{\partial x}\langle p, v\rangle + \frac{\partial}{\partial x}\mathcal{L}(x,v)$.

$$= -\langle p, \frac{\partial v}{\partial x}\rangle + \frac{\partial \mathcal{L}}{\partial x} + \langle \underbrace{\frac{\partial \mathcal{L}}{\partial v}}_{p}, \frac{\partial v}{\partial x}\rangle$$

$$= \frac{\partial \mathcal{L}}{\partial x}$$

So hamilton's second equation is $\dot{p} = \frac{\partial \mathcal{L}}{\partial x}$

$$\dot{p} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial v}$$

which is condition (ii). ■

Basic Variational Calculus

wrt any atlas by adapted coordinates

We proved $\gamma:[a,b] \to Q$ satisfies the Euler-Lagrange equations $\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial v} = \frac{\partial \mathcal{L}}{\partial x}$ iff γ is stationary for

$$A_\gamma = \int_a^b \mathcal{L}(\gamma, \dot{\gamma})\,dt$$

with respect to "short" variations with fixed endpoints.

We claimed: $\gamma:[a,b] \to Q$ is stationary for A_γ is stationary wrt short variations $\bar{w}$ fixed endpoints $\iff$ γ is stationary wrt all variations $\bar{w}$ fixed endpoints.

Proof. Let $\gamma_\epsilon(t)$ be an arbitrary variation of $\gamma(t)$ with fixed endpoints

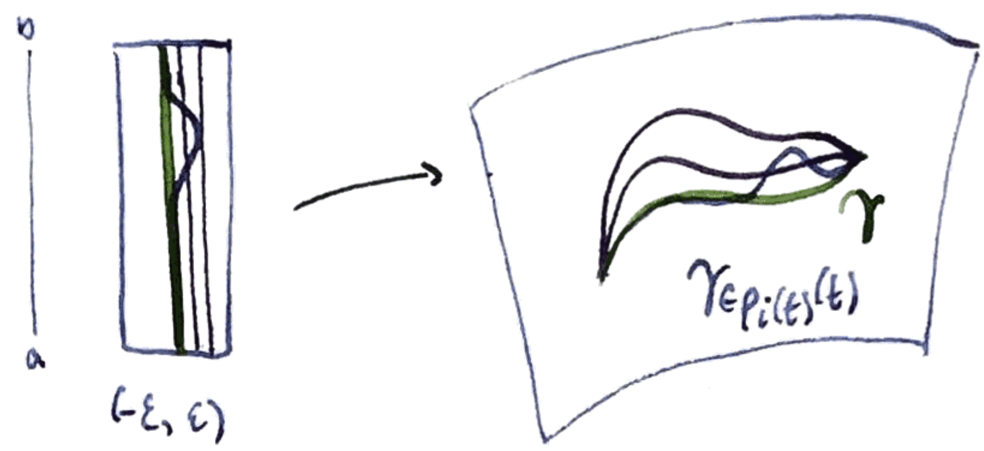
$$0 \overset{\text{WTS}}{\underset{???}{=}} \frac{d}{d\epsilon}\Big|_{\epsilon=0} A_{\gamma_\epsilon} = \int_a^b \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L}(\gamma_\epsilon(t), \dot{\gamma}_\epsilon(t))\,dt$$

Let $p_i:[a,b] \to \mathbb{R}_{\geq 0}$ be a partition of unity (so $\sum_i p_i \equiv 1$) with "small" support (i.e. $\forall i$, $\overline{\{t : p_i(t) \neq 0\}} =: I$ satisfies $\gamma(I) \subset$ domain of a chart of Q).

small deformation

Proof... Then $\gamma_\epsilon^{(i)}(t) := \gamma_{\epsilon p_i(t)}(t)$. We claim:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} A_{\gamma_\epsilon} = \sum_i \frac{d}{d\epsilon}\Big|_{\epsilon=0} A_{\gamma_\epsilon^{(i)}}$$



$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} A_{\gamma_\epsilon} = \int_a^b \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L}(\gamma_\epsilon(t), \dot\gamma_\epsilon(t))\,dt$$

$$= \int_a^b \begin{pmatrix}\text{The derivative of } \mathcal{L} \\ \text{at the point } (\gamma(t), \dot\gamma(t)) \\ \text{in the direction } u(t)\end{pmatrix} dt \quad \text{where } u(t) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\gamma_\epsilon(t), \dot\gamma_\epsilon(t)) \in T_{(\gamma(t), \dot\gamma(t))} TQ$$

Contrast this with $\frac{d}{dt}(\gamma(t), \underbrace{\delta\gamma(t)}_{\frac{d}{d\epsilon}|_{\epsilon=0}\gamma_\epsilon(t)}) \in T_{(\gamma(t), \dot\gamma(t))} TQ$

$$\sum_i \frac{d}{d\epsilon}\Big|_{\epsilon=0} A_{\gamma_\epsilon^{(i)}} = \int_a^b \begin{pmatrix}\text{derivative of } \mathcal{L} \text{ at} \\ (\gamma(t), \dot\gamma(t)) \text{ in the} \\ \text{direction } u^{(i)}(t)\end{pmatrix} dt$$

where $u^{(i)}(t) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\gamma_\epsilon^{(i)}(t), \dot\gamma_\epsilon^{(i)}(t))$

$\boxed{\text{Claim}}$ $\boxed{u(t) = \sum u^{(i)}(t) \ \forall t}$ Proof. A calculation.

WARNING: $u^{(i)}(t) = p_i(t) u(t) + \underbrace{(\text{contribution involving } p_i'(t))}_{\text{cancel out in the sum.}}$