Thm (Noether)

$\boxed{\text{symmetry} \Rightarrow \text{conservation law}}$

Lie group

Given a hamiltonian dynamical system (M, ω, H) and a symmetry $G \circlearrowright M$
$\forall g \in G \ g^*\omega = \omega$ and $H \circ g = H$. $X \in \mathfrak{g} \rightsquigarrow$ 1-parameter subgroup
$\psi_t : M \to M$, given by action by $\exp(tX)$. Assume the flow ψ_t is hamiltonian.
generated by $f \in C^\infty(M)$. ("conjugate momentum" of the symmetry)

e.g. $f = \mu^X = \langle \mu, X \rangle$ for a momentum map $\mu : M \to \mathfrak{g}^*$. Then μ is conserved under the time evolution of H.

Office Hour

$\varphi : N \to M$, $\varphi_*[\underbrace{\sum m_j \sigma_j}_{c}] = \sum m_j (\varphi \circ \sigma_j)$, given $[\alpha] \in H^k_{dR}(M)$,

$$\langle [\alpha], \varphi_*(c) \rangle = \sum m_j \int_{\Delta^k} (\varphi \circ \sigma_j)^* \alpha = \sum m_j \int_{\Delta^k} \sigma_j^*(\varphi^*(\alpha)) = \langle \varphi^*(\alpha), c \rangle$$

Fact: $\exists!$ (up to boundary) smooth cycle $c \in H_{top}(M)$ s.t. $\int_c = \int_M$ called the fundamental class

Example of a momentum map

Consider a collection of N particles in $\mathbb{R}^3$,

$Q = (\mathbb{R}^3)^N$, $q = (x, y, z)$, $p = (p_x, p_y, p_z) = m(\dot{x}, \dot{y}, \dot{z})$
$M = T^*Q = (\mathbb{R}^6)^N = ((\mathbb{R}^3)^N)^2$, $G = SO(3)$, $\mathfrak{g} = \mathbb{R}^3$ w generators
$\sum^N y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $\sum^N z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, $\sum^N x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ for $SO(3) \circlearrowright Q$.

↳ $SO(3) \circlearrowright T^*Q$ is the cotangent lift of $SO(3) \circlearrowright Q$.

↳ Tautological 1-form: $\alpha = \sum^N p_x dx + p_y dy + p_z dz$

↳ Preserved under any cotangent lift of any diffeo. of $Q \Rightarrow$ preserved under $SO(3)$.

Note: $X \in \mathfrak{g} \rightsquigarrow$ v.fields X_Q on Q and X_M on $M = T^*Q$

$$\begin{array}{c} SO(3) \circlearrowright T^*Q \\ \downarrow \pi \\ SO(3) \circlearrowright Q \end{array}$$

$\pi_* X_M = X_Q$ at each point

So $\alpha(X_M) = \underbrace{\sum^N y p_z - z p_y}_{1^{st} \text{ basis element}}, \underbrace{\sum^N z p_x - x p_z}_{2^{nd}}, \underbrace{\sum^N x p_y - y p_x}_{3^{rd}}$

$= $ coordinates of $\sum^N \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = $ total angular momentum

Note: If $\omega = -d\alpha$, then $\mu : M \to \mathfrak{g}^*$ defined by $\mu^X = \alpha(X_M) \ \forall X \in \mathfrak{g}$ is a momentum map
for α G-invariant

Proof. $\underset{\alpha\text{ invariant}}{0} = \underset{\text{Lie derivative}}{\mathcal{L}_{X_M}} \alpha = d\underbrace{\iota_{X_M}\alpha}_{\mu^X} + \iota_{X_M}\underbrace{d\alpha}_{-\omega} \Rightarrow \boxed{d\mu^X = \iota_{X_M}\omega}$ ■

Shortcut to Hamilton's equations

Say a mass m particle moves in $\mathbb{R}^3$ ← configuration space with coordinates $q=(q_1,q_2,q_3)$. More generally and with yet another change of notation: n particles with masses $m_i>0$ and positions $q_i \in \mathbb{R}^3$, $i=1,\dots,n$, subject to a potential $\mathcal{U} \in C^\infty(Q)$, $Q=(\mathbb{R}^3)^n$

Newton's equations: $\underbrace{m_i \frac{d^2 q_i}{dt^2}}_{\text{acceleration}} = \underbrace{-\overset{\text{gradient}}{\nabla}\mathcal{U}(q)}_{\text{force}}$ are what trajectories $q(t)=(q_i(t))_{i=1}^n$ satisfy

Momentum coordinates: $p_i = m_i \frac{dq_i}{dt} \in \mathbb{R}^3$

Energy function: $H(p,q) = \underbrace{\sum_{i=1}^n \frac{1}{2m_i}\|p_i\|^2}_{\text{kinetic energy}} \left(= \sum \frac{1}{2} m_i \|\dot{q}_i\|^2\right) + \underbrace{\mathcal{U}(q)}_{\text{potential energy}}$.

"Phase phace"
—Yael

Phase space: $((\mathbb{R}^3)^n)^2 = T^*(\mathbb{R}^3)^n$ coordinates $q_i, p_i \in \mathbb{R}^3$, $i=1,\dots,n$

Newton's law in $(\mathbb{R}^3)^n \Longleftrightarrow$ Hamilton's equations in $(\mathbb{R}^6)^n$:

$$\left.\begin{aligned} -\frac{\partial H}{\partial q_i} &= -\frac{\partial \mathcal{U}}{\partial q_i} \underset{\text{Newton}}{=} m_i \frac{d^2 q_i}{dt^2} = \frac{dp_i}{dt} \\ \frac{\partial H}{\partial p_i} &= \frac{1}{m_i} p_i \underset{\text{by def. of } p_i}{=} \frac{dq_i}{dt} \end{aligned}\right\} \text{each equation in } \mathbb{R}^3$$

Kinematics

configuration of a system = list of positions of particles in the system

configuration ~~system~~space = {all possible configurations} = Q (smooth manifold)

state of a system = list of positions and velocities

(velocity) phase space = {all possible states} = TQ ← called a holonomic system

Nonexample: ball rolling on a table
$Q = SO(3) \times \mathbb{R}^2$ but velocity phase space $= E \underset{\neq}{\overset{\text{subbundle}}{\subseteq}} TQ$ Here $\mathbb{R}^2$ velocity is determined by $SO(3)$ velocity

Examples

	Q	TQ
n noncolliding particles	$(\mathbb{R}^3)^n \setminus$ diagonals	$(\mathbb{R}^3)^n \setminus$ diagonals $\times (\mathbb{R}^3)^n$
Planar pendulum	S^1	$TS^1 \cong S^1 \times \mathbb{R}$
Spherical pendulum	S^2	$TS^2 \not\cong S^2 \times \mathbb{R}^2$
Rigid body rotating about a fixed point	$SO(3)$	$TSO(3) \cong SO(3) \times \mathbb{R}^3$
rigid body	$SO(3) \ltimes \mathbb{R}^3 = SE(3)$	$TSE(3) \cong SE(3) \times \mathbb{R}^6$

"This is a famous theorem about how one cannot comb a dog" —Yael

Dynamics

Newton's equation: $\underset{\text{given}}{\text{force}}$ = mass · acceleration 2^{nd} order ODE on Q $\rightsquigarrow$ 1^{st} order on TQ

i.e. a v.f. on TQ.

e.g. Gravitation: $m_i \ddot{x}_i = -\sum_{j \neq i} \frac{m_i m_j (x_i - x_j)}{\|x_i - x_j\|^2}$

Lagrangian mechanics : Motion is determined by the Lagrangian $\mathcal{L}: TQ \to \mathbb{R}$

action of the path: $A_\gamma = \int_a^b \mathcal{L}\, dt$, $\mathcal{L} = \mathcal{L}(\gamma(t), \dot{\gamma}(t))$

Hamilton's principle of least action:

The physical path is stationary for the action among all paths in Q with the same endpoints.

Variation of γ w/ fixed endpoints : $\gamma_\epsilon : [a,b] \to Q$, $\epsilon \geq 0$, smooth family $(\epsilon, t) \mapsto \gamma_\epsilon(t)$ smooth

s.t. $\gamma_0(t) = \gamma(t)$, $\forall \epsilon$ $\gamma_\epsilon(a) = \gamma(a)$, $\gamma_\epsilon(b) = \gamma(b)$.

Defn. γ is stationary if $\forall \gamma_\epsilon$ $\frac{d}{d\epsilon} A_{\gamma_\epsilon} = 0$.

Calculus of variations

γ is $\underbrace{\text{stationary}}_{\text{global}}$ for A_γ iff γ satisfies the $\underbrace{\text{Euler Lagrange}}_{\text{local, even infinitesimal}}$ equations
(requires adapted coords)

$\mathcal{L} = \mathcal{L}(x, v)$, $\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = \frac{\partial \mathcal{L}}{\partial x}}$ @ $(\gamma(t), \dot{\gamma}(t))$

Fix $\gamma:[a,b]\to Q$

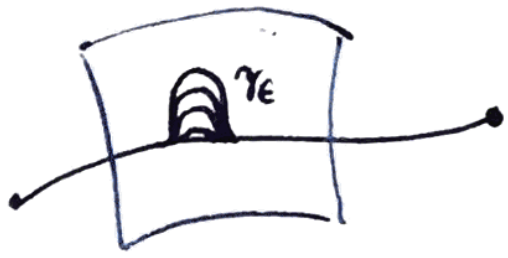
Tentative defn. A subinterval $I\subset[a,b]$ is short if $\gamma(I)\subseteq$ domain of a chart

A variation $\{\gamma_\epsilon\}_\epsilon$ is short if its support $:=\overline{\{t \mid \exists\epsilon\ \gamma_\epsilon(t)\neq\gamma(t)\}}\subset$ short subinterval

Claim

γ stationary for A_γ under all variations with fixed endpoints	$\Leftrightarrow$	γ is stationary for A_γ under all short variations with fixed endpoints	$\Leftrightarrow$	γ satisfies Euler-Lagrange in each adapted coordinate chart.

Proof. $\left.\dfrac{d}{d\epsilon}\right|_{\epsilon=0}\displaystyle\int_a^b \mathcal{L}(x,v)\,dt$ $\qquad x=x(\epsilon,t)=\gamma_\epsilon(t)$, $\quad v=v(\epsilon,t)=\dot\gamma_\epsilon(t)$



$$=\int_a^b \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\mathcal{L}(x,v)\,dt=\int_a^b\left(\sum_j \frac{\partial\mathcal{L}}{\partial x_j}\left.\frac{\partial x_j}{\partial\epsilon}\right|_{\epsilon=0}+\frac{\partial\mathcal{L}}{\partial v_j}\underbrace{\left.\frac{\partial v_j}{\partial\epsilon}\right|_{\epsilon=0}}_{\frac{\partial}{\partial\epsilon}\frac{\partial x_j}{\partial t}=\frac{\partial}{\partial t}\frac{\partial x_j}{\partial\epsilon}}\right)dt$$

$$\rightsquigarrow\int_a^b\frac{\partial\mathcal{L}}{\partial v_j}\cdot\left(\frac{\partial}{\partial t}\frac{\partial x_j}{\partial\epsilon}\right)dt=\underbrace{\left.\frac{\partial\mathcal{L}}{\partial v_j}\frac{\partial x_j}{\partial\epsilon}\right|_a^b}_{0}-\int_a^b\left(\left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v_j}\right)\frac{\partial x_j}{\partial\epsilon}\right)dt$$

integration by parts

$$\text{So }\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}A_{\gamma_\epsilon}=\int_a^b\sum_j\left(\frac{\partial\mathcal{L}}{\partial x_j}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v_j}\right)\cdot\underbrace{\frac{\partial x_j}{\partial\epsilon}}_{\text{arbitrary functions w/ fixed endpoints}}dt$$

This is 0 for all variations $\Leftrightarrow$ $\dfrac{\partial\mathcal{L}}{\partial x_j}=\dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial v_j}$

Legendre transform: Lagrangian formulation $\rightsquigarrow$ hamiltonian formalism

Overview: given $\mathcal{L}(x,v): TQ\to\mathbb{R}$, we produce a map $\Psi: TQ\to T^*Q$, $(x,v)\mapsto(x,p)$
in adapted coordinates $p=\dfrac{\partial\mathcal{L}}{\partial v}$. w/o coordinates $f:=\mathcal{L}|_{T_xQ}: T_xQ\to\mathbb{R}$
$\forall v\in T_xQ$ $df|_v\in T_x^*Q$, take $\Psi(x,v)=(x,p)$ where $p=df|_v$. If $\Psi: TQ\xrightarrow{\sim}T^*Q$,
produce a function $H: T^*Q\to\mathbb{R}$, $H=\langle \underset{T_x^*Q}{p},\underset{T_xQ}{v}\rangle-\mathcal{L}(x,v)$ w/ $v=v(x,p)$
via $(x,p)\overset{\Psi'}{\longmapsto}(x,v)$

Thm

> $\gamma:[a,b]\to Q$ satisfies Euler-Lagrange equations for $\mathcal{L}$ iff $\Psi_0(\gamma,\dot\gamma):[a,b]\to T^*Q$ satisfies Hamilton's eqn's for H.
> Moreover, every solution of hamilton's equations has this form.