Then Ψ is a ham. isot. from ψ^1 to $\tilde{\psi}^2$ gen. by:

$$\tilde{H}_\tau = \begin{cases} H^1_\tau & \tau \in [0,1] \\ H^2_{\tau-1} & \tau \in [1,2] \end{cases}$$

$Ham(M,\omega)$ is a group. Let $\psi, \varphi \in Ham(M,\omega)$

$\{\psi_t\}_{t\in[0,1]}$ $\psi_0 = Id$, $\psi_1 = \psi$
$\{\varphi_t\}_{t\in[0,1]}$ $\varphi_0 = Id$ $\varphi_1 = \varphi$
} gen by ham's that vanish near $t=0$ & $t=1$.

Then $\tilde{\psi}_t := \begin{cases} \psi_\tau & , \tau \in [0,1] \\ \varphi_{\tau-1} \circ \psi, & \tau \in [1,2] \end{cases}$ is a ham. isot. from Id to $\varphi \circ \psi$

and $\{\varphi_{1-t} \circ \varphi^{-1}\}$ is a ham. isot. from Id to φ^{-1}.

~~Then $\tilde{\Psi}$ is a ham. isot.~~

Dual of a Lie algebra

Recall: A Poisson bracket is $\{\cdot,\cdot\}: C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ which
- ↳ is a Lie bracket.
- ↳ satisfies the Leibniz rule.

E.g. $M = \mathfrak{g}^*$

For G a Lie group and $\mathfrak{g} = T_eG$. given $X, Y \in \mathfrak{g}$, we have $a: \mathbb{R} \longrightarrow G$, $t \longmapsto a_t$, $0 \longmapsto e$

such that $\frac{d}{dt}\big|_{t=0} a_t = X$. Then $\boxed{[X,Y] = \frac{d}{dt}\Big|_{t=0} Ad(a_t)(Y)}$

On $\mathfrak{g}^*$ the dual, $f \in C^\infty(\mathfrak{g}^*)$ ⇝ $\forall \beta \in \mathfrak{g}^*$ $df|_\beta \in T^*_\beta \mathfrak{g}^* \overset{\text{canonical}}{\cong} \mathfrak{g}$

⇝ $\{f,g\}(\beta) = \langle \underbrace{\beta}_{\in \mathfrak{g}^*}, \underbrace{[df|_\beta, dh|_\beta]}_{\in \mathfrak{g}} \rangle$

$T^*_\beta \mathfrak{g}^* = (T_\beta \mathfrak{g}^*)^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$

Kirillov-Kostant-Souriau: Symplectic structure on coadjoint orbits

$Ad^*: G \circlearrowright \mathfrak{g}^*$. For $X \in \mathfrak{g}$, we get $X^\#$ a vector field on $\mathfrak{g}^*$:

$\forall \beta \in \mathfrak{g}^*$, $X^\#|_\beta \in T_\beta \mathfrak{g}^* \cong \mathfrak{g}^*$ (so $X^\#$ is just a function on $\mathfrak{g}^* \to \mathfrak{g}^*$).

$\forall Y \in \mathfrak{g}$, $\boxed{\langle X^\#|_\beta, Y\rangle := \langle \beta, [Y, X]\rangle}$ (up to $\pm$).

Given $\lambda \in \mathfrak{g}^*$, let $\mathcal{O} = \mathcal{O}_\lambda := Ad^*(G)(\lambda)$, the coadjoint orbit through λ.

$\forall \beta \in \mathcal{O}$ $\{X^\#|_\beta : X \in \mathfrak{g}\} = T_\beta \mathcal{O}$. We put a symplectic structure on this:

$$\boxed{\omega_{KKS}|_\beta(X^\#|_\beta, Y^\#|_\beta) := \langle \beta, [X, Y]\rangle}$$

→ well-defined
→ nondegenerate
→ closed

} easy theorem

$\rightsquigarrow (\mathcal{O}, \omega_{KKS})$ is symplectic.

For $\mathcal{O} \subset \mathfrak{g}^*$ a coadjoint orbit, $f, h \in C^\infty(\mathfrak{g}^*)$. Then

Easy Theorem

$$\{f, h\}|_\mathcal{O} = \{f|_\mathcal{O}, g|_\mathcal{O}\}$$

in $\mathfrak{g}^*$ wrt ω_{KKS}

E.g. $G = SU(2)$, $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathbb{R}^3$ (noncanonically) $Ad^*: G \circlearrowright \mathfrak{g}^* \cong \mathbb{R}^3$ via $SU(2) \xrightarrow{2:1} SO(3)$. On each sphere, $\omega_{KKS} = \dfrac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{x^2+y^2+z^2}$

We have a coadjoint action $G \circlearrowright (\mathcal{O}, \omega_{KKS})$. Then the inclusion $\mu: \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is a momentum map (up to $\pm$). Its coordinates are $\forall X \in \mathfrak{g}$

$$\mu^X: \mathcal{O} \longrightarrow \mathbb{R}$$
$$\beta \longmapsto \langle \beta, X\rangle$$

$d\mu^X = \iota_{X^\#}\omega_{KKS}$ and $\mu: \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is equivariant.

Back to Hamiltonian Mechanics...

Evolution of observables

$H \in C^\infty(M, \omega) \rightsquigarrow X_H \rightsquigarrow \varphi_t^H : M \to M$ time evolution.

$\forall f \in C^\infty(M)$ (an observable), $\frac{d}{dt}\big|_{t=0} (f \circ \varphi_t^H) = X_H f = \{f, H\}$ derivative

f is conserved under the flow of $H \iff \{f, H\} = 0 \iff H$ is conserved under the flow of $f \iff$ the flow of f is a symmetry of (M, ω, H).

Thm (Noether)

$\boxed{\text{symmetry} \Rightarrow \text{conservation law}}$

Lie group

Given a hamiltonian dynamical system (M, ω, H) and a symmetry $G \circlearrowright M$ $\forall g \in G$ $g^*\omega = \omega$ and $H \circ g = H$. $X \in \mathfrak{g} \rightsquigarrow$ 1-parameter subgroup $\psi_t : M \to M$, given by action by $\exp(tX)$. Assume the flow ψ_t is hamiltonian. generated by $f \in C^\infty(M)$. ("conjugate momentum" of the symmetry)

e.g. $f = \mu^X = \langle \mu, X \rangle$ for a momentum map $\mu : M \to \mathfrak{g}^*$. Then μ is conserved under the time evolution of H.

Office Hour

$\varphi : N \to M$, $\varphi_*[\underbrace{\sum m_j \sigma_j}_{c}] = \sum m_j (\varphi \circ \sigma_j)$, given $[\alpha] \in H^k_{dR}(M)$,

$$\langle [\alpha], \varphi_*(c) \rangle = \sum m_j \int_{\Delta^k} (\varphi \circ \sigma_j)^* \alpha = \sum m_j \int_{\Delta^k} \sigma_j^*(\varphi^*(\alpha)) = \langle \varphi^*(\alpha), c \rangle$$

Fact: $\exists!$ (up to boundary) smooth cycle $c \in H^{top}_{top}(M)$ s.t. $\int_c = \int_M$ Called the fundamental class