Hamiltonian Dynamics

[Previous lecture in research notebook]

Given (M,ω) symplectic with $H \in C^\infty(M)$ the hamiltonian, inducing a v.f. X_H such that

$$\boxed{dH = \iota_{X_H}\omega} \quad \text{Hamilton's equation}$$

The flow $\psi_t = \psi_t^H : M \longrightarrow M$, $t \in \mathbb{R}$

$\frac{d}{dt}\psi_t = X_H \circ \psi_t$ (defined $\forall t \in \mathbb{R}$ if, e.g., H is compactly supported)

$\psi_{t+s} = \psi_t \circ \psi_s$

$\mathcal{L}_{X_H} H = 0$, so $H \circ \psi_t = H$ (flows live in level sets of H).

$\mathcal{L}_{X_H}\omega = 0$, so $\psi_t^*\omega = \omega$ (ψ_t is symplectic),

and $\psi_t^*\left(\frac{\omega^n}{n!}\right) = \frac{\omega^n}{n!}$ (ψ_t is measure-preserving).

Geometry on a regular energy level set

$H : M \to \mathbb{R} \ni c$ reg. value

$\forall p \in H^{-1}(c)$, $dH|_p : T_pM \to \mathbb{R}$ is onto

By implicit function thm, $Z := H^{-1}(c) \subset M$
is an embedded submanifold of codimension 1.

Null directions of $\omega|_{T_pZ} = T_pZ \cap \underbrace{(T_pZ)^{\omega}}_{\dim 1}$

must be nonzero because $\dim Z$ is odd.
Contains $\mathbb{R}.X_H|_p$, so it is equal to it. $\forall p \in Z$.

$X_H \in \underbrace{TZ}_{\ker dH}$ $\quad X_H \neq 0$, since $\imath_{X_H}\omega = dH \neq 0$

along Z (by restriction) on $TM|_Z$.

X_H has no friends in Z:

$\forall \zeta \in TZ$, $\omega(X_H, \zeta) = dH(\zeta) = 0$

$\zeta \in TZ = \ker dH$

On $Z = H^{-1}(c)$:

Defn. $\mathbb{R}.X_H = \text{null}(\omega|_{TZ})$ is the (rank one) **null foliation**, or the **characteristic foliation** of Z.

If Z is a regular level set of H and of $\hat{H}$, inducing the same coorientation, then $X_{\hat{H}} = X_H$
(a positive function).

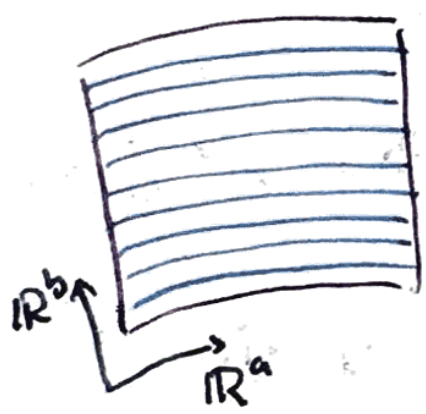
Recall / defn: For Z a manifold, a <u>distribution</u> on Z is a subbundle E of TZ.

Further, it is a <u>foliation</u> if near each point of Z $\exists$ a chart

$$\begin{array}{ccc} Z & & \mathbb{R}^a \times \mathbb{R}^b \\ \cup & & \cup \\ U & \xrightarrow{\varphi} & \Omega \end{array}$$

such that at each point x

$$\varphi_* E_x = \mathbb{R}^a \times \{0\}$$



Thm (Frobenius) A distribution E is a foliation iff E is <u>involutive</u>: i.e. $\forall$ vector fields $\xi, \eta \in \Gamma(E)$, their bracket is also in E: $[\xi, \eta] \in \Gamma(E)$.

Such an E is called <u>integrable</u>.

E.g. rank $E = 1$ $\Rightarrow$ locally $E = \mathbb{R} \cdot \xi$ for a nonvanishing field $\xi \Rightarrow$ integrable: locally $\exists$ coordinates $x_1, \ldots, x_m$ such that $\xi = \frac{\partial}{\partial x_1}$

E.g. Let ω be a closed 2-form on Z of constant rank (i.e. rank $\omega^\#: TZ \to T^*Z$ is constant) $\iff$ $\dim \ker \omega^\#$ is constant. Then $E := \text{null}\,\omega$ is a foliation.

Proof.

Note: For any 2-form ω and v.f.'s a, b, c,

$$(d\omega)(a,b,c) = \underbrace{a\,\omega(b,c)}_{\substack{\text{derivative of} \\ \omega(b,c) \text{ in the} \\ \text{direction } a}} + b\,\omega(c,a) + c\,\omega(a,b)$$

$$- \big(\omega([a,b],c) + \omega([b,c],a) + \omega([c,a],b)\big)$$

Suppose $d\omega = 0$. Let a,b,c be v.f.'s in null ω.

Goal: $[a,b] \in \text{null}\,\omega$ Take any v.f. c:

$$0 = (d\omega)(a,b,c)$$

$$= a\,\cancelto{0}{\omega(b,c)} + b\,\cancelto{0}{\omega(c,a)} + c\,\cancelto{0}{\omega(a,b)}$$

$$- \omega([a,b],c) - \cancelto{0}{\omega([b,c],a)} - \cancelto{0}{\omega([c,a],b)}$$

So $[a,b] \in \ker \omega^\#$.

E.g. $H : \mathbb{R}^{2n} \to \mathbb{R}$

$$H(x_1, y_1, \ldots, x_n, y_n) = x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2$$

Write it as $H(z) = |z|^2$

Then $H^{-1}(1)$ = the unit sphere in $\mathbb{R}^{2n} = \mathbb{C}^n$

Then the flow of H is (up to scalar) the (Hopf) S^1-action $t : z \mapsto e^{it} \cdot z$

all trajectories are periodic.

Weinstein $M = \mathbb{R}^{2n}$ $\{H \leq c\}$ is compact and convex. Then $\{H = c\}$ has a periodic trajectory iff it has closed characteristic.

More generally,

Conj. (Weinstein) Z compact manifold of dim $2n-1$. e.g. $H^{-1}(c)$ as before. α a contact 1-form (i.e. $\alpha \wedge (d\alpha)^{\wedge n-1} \neq 0$), e.g. α st. $d\alpha = \omega$ as before, then null $d\alpha$ (which has rank 1) has a closed characteristic).

Office Hour

Q6.2 For f an orientation-preserving diffeomorphism

$$f : M \to M'$$

Then $[f^*\alpha]\cdot[f^*\beta] = [\alpha]\cdot[\beta]$.

$A \in GL_n\mathbb{Z} \iff A : \mathbb{R}^n \to \mathbb{R}^n$ takes $\mathbb{Z}^n$ to $\mathbb{Z}^n$

$\iff A$ has $\mathbb{Z}$ entries & $\det A = \pm 1$.

Case: $n=2$

$$\begin{array}{ccccc} \mathbb{R}^2/\mathbb{Z}^2 & \longrightarrow & \mathbb{C}/\Gamma & & \\ e_1 & \longmapsto & u & \text{or} & 1 \\ e_2 & \longmapsto & v & & \tau \end{array}$$

By above, $GL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n$ fixing $\mathbb{Z}^n$, so

$$GL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$$
$$\parallel$$
$$\mathrm{Aut}(\mathbb{R}^n/\mathbb{Z}^n)$$

$\psi : A \to B$

e.g. if B is metrizable

$$\boxed{\psi \text{ proper}} \;\underset{\text{always!}}{\overset{\text{if } B \text{ is cpctly gen}}{\rightleftarrows}}\; \boxed{\begin{array}{c}\psi \text{ closed} \\ \forall b\ \psi^{-1}(b) \text{ compact}\end{array}}$$