Hermitian Structures

Defn. The standard Hermitian structure on $\mathbb{C}^n = \mathbb{R}^{2n}$

$$H(u,v) = \sum_{j=1}^{n} \bar{u}_j v_j$$

$$= g(u,v) + i\omega(u,v)$$

with g the standard inner product on $\mathbb{R}^{2n}$ and ω the standard symplectic tensor on $\mathbb{R}^{2n}$.

In general, a Hermitian structure is a nondegenerate, sesquilinear form $H : \underset{\text{skew-linear}}{V} \times \underset{\text{linear}}{V} \to \mathbb{C}$

such that $H(u,v) = \overline{H(v,u)}$ and $\forall u \neq 0$, $H(u,u) > 0$

Fact H is a Hermetian structure if and only if $H = g + i\omega$ for some inner product g, and symplectic tensor ω on V as $\mathbb{R}$-v.s. such that

$J : V \to V$, $v \mapsto iv$ satisfies $\boxed{g(u,v) = \omega(u, Jv)}$

Exercise: All Hermitian structures are isomorphic to the standard $\mathbb{C}^n$.

There is no time. Let me sum up:

- On a real vector space V, consider
  - g an inner product
  - ω a symplectic tensor
  - $J : V \to V$, $J^2 = -\mathbb{1}$ a complex structure

The three are compatible if

$$\boxed{g(u,v) = \omega(u, Jv)} \quad (*)$$

It follows any two of g, ω, J determine the third.

Defn. g, ω are compatible if $\exists J$ st. $J^2 = -\mathbb{1}$ and $(*)$

Defn. J, ω are compatible if ~~$\exists g$~~ $(u,v) \mapsto \omega(u, Jv)$ is symmetric and positive-definite.

Q. What does compatibility between g and J look like?

Defn. Given a symplectic vector space (V, ω),

$$\mathcal{J}(V,\omega) := \{\text{compatible } J\}$$

$$\mathcal{G}(V,\omega) := \{\text{compatible } g\}$$

We have: a bijection $\mathcal{J}(V,\omega) \leftrightarrow \mathcal{G}(V,\omega)$

<u>Indeed</u>: $\mathcal{J}$ and $\mathcal{G}$ are smooth manifolds diffeo. to some $\mathbb{R}^m$, and the bijection is $Sp(V,\omega)$-equivariant, where:

<u>Defn</u>. Given V and symplectic g, ω, J:

$$Sp(V,\omega) := \{A \in Aut_{\mathbb{R}}(V) : \omega(Au, Av) = \omega(u,v)\}$$

$$GL_{\mathbb{C}}(V) := Aut_{\mathbb{C}}(V) = Aut_{\mathbb{R}}(V) \cap \ker(ad_J)$$

$$O(V) := \{A \in Aut_{\mathbb{R}}(V) : \langle Au, Av\rangle = \langle u,v\rangle\}$$

The intersection of any two is:

$$U(V) = \{A \in GL_{\mathbb{R}}(V) : H(u,v) = H(Au, Av)\}$$

where $H = g + i\omega$.

<u>Remark</u>: Through the hom-tensor adjunction and either a metric or symplectic tensor, we can move between bilinear forms and linear maps. These are both represented by matrices.

Notation (or Lie lies): In Lie theory the notation:

$$Sp(2n) = \mathrm{Aut}(\mathbb{C}^{2n}, \omega \otimes \mathbb{C}) \cap \mathcal{U}(2n)$$

We will **not** use this notation.

Note: For $\dim V = 2n$,

$$\mathfrak{g}(V,\omega) \subset \hom(V,V) \cong \mathbb{R}^{(2n)^2}$$
$$\mathcal{G}(V,\omega) \subset (V \otimes V)^* \cong \mathbb{R}^{(2n)^2}$$

Crucial Lemma For any symplectic v.s. (V,ω), there exists a retraction

$$\pi : \left\{ \begin{matrix} \text{inner products} \\ \text{on } V \end{matrix} \right\} \longrightarrow \mathcal{G}(V,\omega)$$

Corollary $\mathcal{G}(V,\omega)$ is contractible.

Proof. The space of inner products is convex. Given $g, g_0 \in \mathcal{G}(V,\omega)$, the image of the line segment connecting g and g_0 under π gives a path b/w g and g_0 in $\mathcal{G}(V,\omega)$.

Proof (crucial lemma). Given g, let A be defined by

$$g(u,v) = \omega(u, Av)$$

Let A^T be the transpose of A w.r.t. g. Then A^TA is symmetric and positive def. wrt g. Hence for all $\alpha \in \mathbb{R}$, we can take $(A^TA)^\alpha$ why?

Note: $C_{End(V)}(A^TA) \subseteq C_{End(V)}((A^TA)^\alpha)$

Let $J = A \cdot (A^TA)^{-\frac{1}{2}}$. Then $J \in \mathcal{J}(V,\omega)$.

Define $\pi(g) = \omega(\cdot, J\cdot)$. ∎

Aside on LinAlg:

Polar decomposition wrt g: $GL_{\mathbb{R}}(V) \underset{g}{\overset{\text{diffeo.}}{\cong}} O(V) \times \left\{ \begin{matrix} \text{symmetric} \\ \text{pos. definite} \end{matrix} \right\}$

$$A \longmapsto \left[A(A^TA)^{-\frac{1}{2}}\right] \cdot (A^TA)^{\frac{1}{2}}$$

Fact A diagonalizable $\iff$ $\exists g$ st. A sym. wrt. g.

- Given a diagonalizable function, it makes sense to apply any $f: \mathbb{C} \to \mathbb{C}$ to the operator by applying it eigenvalue-wise. One may need to be careful with how f behaves near the spectrum of $T \in End(V)$