Area measurement

$$\varphi : \underset{\substack{\uparrow \\ \text{2-mfld} \\ \text{possibly w/ bdry}}}{\Sigma} \longrightarrow \underset{\substack{\uparrow \\ \text{almost} \\ \text{Kähler}}}{(M, \omega, g)}$$

If Σ is oriented $\Longrightarrow$ symplectic area $= \int_\Sigma \varphi^* \omega$

If Σ is noncompact $\Longrightarrow$ the integral is improper

> Recall: A domain of integration is a subset $D \subseteq \Sigma$ such that $\bar{D}$ is compact and ∂D has measure zero. In the improper case we say that
>
> $$\int_\Sigma \varphi^* \omega = I$$
>
> when $\forall\, \varepsilon > 0$ $\exists\, D_0$ a domain of integration such that if $D_0 \subseteq D$ then
>
> $$\left| \int_D \varphi^* \omega - I \right| < \varepsilon$$

Riemannian Area

In local coordinates x_1, x_2 on Σ: $\varphi^* g = \sum_{i,j} A_{ij}\, dx_i \otimes dx_j$

where $A_{ij} = g\left(\varphi_* \frac{\partial}{\partial x_i}, \varphi_* \frac{\partial}{\partial x_j} \right)$. The matrix (A_{ij}) is

symmetric positive semidefinite and positive definite wherever

$d\varphi \neq 0$.

The Riemannian area is defined by

$$\int \sqrt{\det A}\; dx_1\, dx_2$$

Note: This quantity is invariant under reparametrizations.

Lemma: For (oriented) $\varphi : \Sigma \to (M, \omega, g)$

$$\text{symplectic area} \leq \text{Riemannian area}$$

~~with~~ with equality iff $\forall\, x \in \Sigma$: (If $d\varphi|_x$ has rank 2)

i) $d\varphi|_x (T_x \Sigma) \subseteq T_{\varphi(x)} M$ is a complex subspace

ii) ~~$d\varphi|_x \neq 0$ then~~ the given orientation on Σ agrees with the complex orientation from M.

eg. If $\varphi : \Sigma \hookrightarrow M$ is an embedding then we have equality if and only if Σ is an almost complex submanifold.

Proof: It suffices to show this in a oriented coordinate patch, ie, WLOG let $\Sigma \subseteq \mathbb{R}^2$. It also suffices to prove pointwise for $u = \varphi_* \frac{\partial}{\partial x_1}$ & $v = \varphi_* \frac{\partial}{\partial x_2}$ WLOG $u, v \in T_{\varphi(x)} M \cong \mathbb{C}^n$ with the standard Kähler structure.
Now, we claim that

$$\omega(u,v) \leq \sqrt{\langle u,u\rangle \langle v,v\rangle - \langle u,v\rangle^2}$$

with equality iff u & v differ by a complex scalar ~~and~~ with imaginary part ≥ 0. (if $u, v \neq 0$ then $v = \lambda u$ with $\operatorname{im} \lambda \geq 0$).

Indeed, Cauchy-Schwartz, for the Hermitian structure gives

$$|H(u,v)| \le \|u\|^2 \cdot \|v\|^2$$

with equality iff u, v differ by a complex scalar, where

$$H = \langle \cdot, \cdot \rangle + i\,\omega(\cdot, \cdot)$$

Now,

$$\|H(u,v)\|^2 = \langle u,v\rangle^2 + (\omega(u,v))^2,$$

$$\|u\|^2\|v\|^2 = \langle u,u\rangle \cdot \langle v,v\rangle$$

implying the claimed result. ■

<u>Corollary</u>: Given an almost Kähler manifold (M, g, ω) and a holomorphic curve

$$\varphi_0 : \underset{\substack{\text{closed Riemann} \\ \text{surface}}}{\Sigma_0} \longrightarrow M$$

let $A := [\Sigma_0] \; (= [\varphi_0]) \in H_2(M)$. If

$$\varphi : \Sigma \to M$$

is another smooth cycle such that $\overset{[\varphi]}{\overset{\shortparallel}{[\Sigma]}} = A$ then

$$\operatorname{area}_g \Sigma_0 \le \operatorname{area}_g \Sigma$$

Proof:

$$\operatorname{area}_g \Sigma_0 \underset{\substack{\text{since } \varphi_0 \\ \text{is holomorphic}}}{=} \int_{\Sigma_0} \varphi_0^* \omega \underset{[\Sigma_0]=[\Sigma]}{=} \int_{\Sigma} \varphi^* \omega \underset{\substack{\text{Wirtinger's} \\ \text{inequality}}}{\leq} \operatorname{area}_g \Sigma$$

Version w/ boundary:

When $(M, \omega, g) = \mathbb{C}^n$ $\omega = d\alpha$, given compact oriented 2-manifolds with boundary in $\mathbb{C}^n$ such that $\partial C_1 = \partial C_2$ assume that C_1 is complex. Then

$$\operatorname{area}_g C_1 \leq \operatorname{area}_g C_2$$

Proof:

$$\operatorname{area}_g C_2 \underset{(\text{Wirtinger})}{\geq} \int_{C_2} \omega \underset{(\text{Stokes})}{=} \int_{\partial C_2} \alpha = \int_{\partial C_1} \alpha \underset{(\text{Stokes})}{=} \int_{C_1} \omega$$

$$= \operatorname{area}_g C_1$$

where the last equality follows because C_1 is complex. ∎

An application (sources: Stolzenberg 1966 Springer LNM, Hummel: Gromov compactness ...)

Thm: ~~be~~

Let $R > 0$ and $\varphi : N \to B^{2n}(R) \subset \mathbb{C}^n$ be a proper connected nonconstant holomorphic curve through $0 \in \mathbb{C}^n$. Then $\operatorname{area} N \geq \pi R^2$.

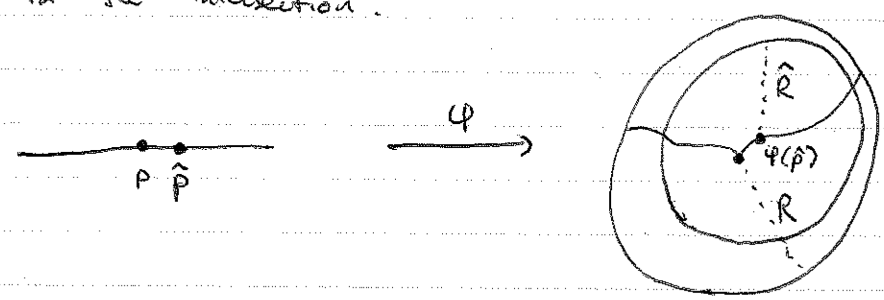
Note: (i) $\operatorname{area} N = \operatorname{area} \varphi(N)$ (ii) φ is holomorphic $\Rightarrow$ $\operatorname{area}_g N = \operatorname{area}_\omega N$.

Proof

We ~~it~~ can reduce this to the following.

($\star$) Let $\hat{R} > 0$ & $\varepsilon > 0$ and let $\hat{\varphi} : \hat{N} \to B^{2n}(\hat{R} + \varepsilon)$ be a proper holomorphic curve such that $\exists\ \hat{p} \in \hat{N}$ with $\hat{\varphi}(\hat{p}) = 0$ & $d\hat{\varphi}|_{\hat{p}} \neq 0$. Then area $\hat{N} \geq \pi \hat{R}^2$.

To see that the above implies the result ~~we~~ assume that $\varphi : N \to B^{2n}(R)$ is as in the statement of the theorem. Let $\varepsilon > 0$ be such that $2\varepsilon < R$. Using the fact that N is connected & φ is nonconstant we use the Taylor expansion to show that points where $d\varphi = 0$ are isolated in N. Thus $\{d\varphi \neq 0\}$ is dense in N so it meets the nonempty open set $\{|\varphi| < \varepsilon\}$. Let $\hat{p} \in N$ be a point in the intersection.



Let $\hat{\varphi} = \varphi - \varphi(\hat{p})$ and $\hat{N} = \hat{\varphi}^{-1}(B^{2n}(R - \varepsilon)) = \varphi^{-1}(\varphi(\hat{p}) + B^{2n}(R - \varepsilon))$

Since $\varphi(\hat{p}) + B^{2n}(R - \varepsilon) \subseteq B^{2n}(R)$ $\hat{\varphi} : \hat{N} \to B^{2n}(R - \varepsilon)$ is proper.

Also, $\hat{\varphi}(\hat{p}) = 0$ & $d\hat{\varphi}|_{\hat{p}} \neq 0$. Then by the above ($\star$) with $\hat{R} = R - 2\varepsilon$ we find that area $\hat{N} \geq \pi (R - 2\varepsilon)^2$

So

$$\underset{(N \supseteq \hat{N})}{\text{area } N \geq \text{area } \hat{N}} \geq \pi (R - 2\varepsilon)^2$$

and ε can be taken arbitrarily small implying the result.

To prove the intermediate result ($\star$) we will prove the following theorem.

<u>Thm</u>: Let $R>0$ and $\varepsilon>0$ and let $\varphi : N \to B^{2n}(R+\varepsilon)$ be a holomorphic curve such that $\exists\ p \in N$ with $\varphi(p) = 0$ and $d\varphi|_p \neq 0$. Futhermore let ~~$N_R := \varphi^{-1}(B^{2n}(R))$~~ $N_r := \varphi^{-1}(B^{2n}(r))$
Then

i) $\liminf\limits_{r \to 0^+} \dfrac{\text{area } N_r}{\pi r^2} \geq 1$

ii) $r \longmapsto \dfrac{\text{area } N_r}{\pi\ r^2}$ is weakly increasing.

Consequently $\dfrac{\text{area } N_r}{\pi r^2} \geq 1$ for all $r \in (0, R]$.

<u>Proof</u>: (ii)

$x \longmapsto \|\varphi(x)\| : N \to [0, R+\varepsilon)$ is smooth outside $\varphi^{-1}(0)$. By Sard's theorem almost every $R \in [0, R+\varepsilon)$ is a regular value. By properness, the set of singular values is closed. Let U be the set of regular values. Then U is open and has full measure.
Now $r \longmapsto \text{area } N_r$ is weakly increasing. We will show that on U $\dfrac{\text{area } N_r}{\pi r^2}$ is differentiable and $\dfrac{d}{dr} \dfrac{\text{area } N_r}{\pi r^2} \geq 0$.

Since U^c is closed and has zero measure this implies that $r \longmapsto \dfrac{\text{area } N_r}{\pi r^2}$ is weakly increasing everywhere.

$\forall$ r regular values of $x \longmapsto \|\varphi(x)\|$

$$\overline{N_r} = \{ x \mid \|\varphi(x)\| \leq r \}$$

is a manifold w/ boundary $\partial \overline{N}_r = \{ x \mid \|\varphi(x)\| = r \} = N_r \cap \partial B^{2n}(r)$

a smooth curve.

For such r $\exists$ $\frac{d}{dr}$, area $N \geq \text{length}_g\ \partial N_r$

so

$$\frac{d}{dr}\ \frac{\text{area } N_r}{\pi r^2} = \frac{1}{\pi r^2}\left(\frac{d}{dr}\ \text{area } N_r\right) - \frac{2}{\pi r^3}\ \text{area } N_r$$

$$= \frac{2}{\pi r^3}\left(\frac{r}{2}\ \frac{d}{dr}\ \text{area } N_r - \text{area } N_r\right)$$

$= \text{area of the cone over } \partial N_r\ (C_r)$

Wirtinger $\Longrightarrow$ area $C_r \geq$ area N_r

Hilroy