Followup on guest lecture

Consider some principal G-bundle P over Σ

$$\begin{tikzcd} \mathcal{G} \arrow[r, phantom, "\circlearrowright"] & P \arrow[d] \arrow[r, phantom, "\circlearrowleft"] & G \\ & \Sigma & \end{tikzcd}$$

gauge group $\uparrow$

E.g. $P = \Sigma \times G$

<u>Note</u>: A right-equivariant diffeomorphism (i.e. invariant under right multiplication) $\Psi : G \to G$ satisfies $\forall a \in G \ \forall g \in G$

$$\Psi(g.a) = \Psi(g).a$$

In particular, $\forall a \in G$, ~~$\Psi(e)$~~

$$\Psi(a) = \Psi(e.a) = \Psi(e).a$$

So Ψ is <u>left translation</u> by an element $(\Psi(e))$ of G.

<u>So</u>, a trivialization $P = \Sigma \times G$ determines an identification of $\mathcal{G}$ with $C^\infty(\Sigma, G)$.

"Punchline" from Lisa's talk:

$$\underset{\text{gauge group}}{\mathcal{G}} \curvearrowright \underset{\text{connections on } P}{\mathcal{A}} \xrightarrow{\mu}$$

$$\nabla \longmapsto \text{curvature of connection}$$

$$\frac{\text{flat connections}}{\text{gauge}} = \frac{\mu^{-1}(0)}{\mathcal{G}}$$

↳ $\mathcal{A}$ is symplectic

↳ $\mathcal{G} \curvearrowright \mathcal{A}$ is Hamiltonian

↳ μ is a momentum map

Defn. $\mathcal{A}/\!/\mathcal{G} := \dfrac{\mu^{-1}(0)}{\mathcal{G}}$ is the reduced space. [Atiyah-Bott 1982]

↑ This is the symplectic quotient

Lie groups & Lie algebras

Let G be a Lie group, $\mathfrak{g} = T_eG$ its Lie algebra with Lie bracket $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$.

$\exists$ exponential map $\exp: \mathfrak{g} \longrightarrow G$ ← only a group hom $(\mathfrak{g},+) \longrightarrow (G,\cdot)$ if G abelian.

$$\forall \xi \in \mathfrak{g} \qquad \begin{array}{c} t \longmapsto \exp(t\xi) \\ \mathbb{R} \longrightarrow G \end{array}$$

is a 1-parameter subgroup and $\left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) = \xi$.

We have an action

$$\mathrm{Ad} : G \circlearrowright G$$

$$\mathrm{Ad}(a)(g) := aga^{-1}$$

which fixes e. Linearize at e (i.e. $\mathrm{Ad}(a) : G \to G$ $\Rightarrow d_e\mathrm{Ad}(a) : \mathfrak{g} \to \mathfrak{g}$) to get

$$\mathrm{Ad} : G \circlearrowright \mathfrak{g}$$

Dualize: $\mathrm{Ad}^* : G \circlearrowright \mathfrak{g}^*$, the coadjoint action: for $\varphi \in \mathfrak{g}^*$

$$(\mathrm{Ad}^*(a)\,\varphi)\,\xi := \varphi(\mathrm{Ad}(a^{-1})\,\xi)$$

That is: $\mathrm{Ad}^*(a) = (\mathrm{Ad}(a^{-1}))^*$

Next, given $\mathrm{Ad} : G \circlearrowright \mathfrak{g}$, linearize at e again

$$\left(\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}) \rightsquigarrow \underset{\substack{\shortparallel \\ \mathrm{ad}}}{d_e\mathrm{Ad}} : \mathfrak{g} \to \underset{\substack{\cong \\ \mathrm{End}(\mathfrak{g})}}{T_{id}\,\mathrm{End}(\mathfrak{g})}\right)$$

So $\mathrm{ad} : \mathfrak{g} \circlearrowright \mathfrak{g}$ defines the Lie bracket:

$$[A,B] := \mathrm{ad}(A)(B)$$

For matrix groups:

$$[A,B] = AB - BA$$

$$\mathrm{Ad}(A)(b) = AbA^{-1}$$

$$\exp(A) = \mathbb{I} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

E.g. $G = SU(n) = \{a \in GL_n(\mathbb{C}) \mid aa^* = \mathbb{1},\ \det a = 1\}$

$$\mathfrak{g} = \mathfrak{su}(n) = \{a \in \mathfrak{gl}_n(\mathbb{C}) = Mat_n(\mathbb{C}) \mid \underset{\substack{\uparrow \\ \text{anti-Hermitian}}}{a + a^* = 0},\ \underset{\substack{\uparrow \\ \text{traceless}}}{\operatorname{tr} a = 0}\}$$

Group actions

$G \circlearrowright M$, $\xi \in \mathfrak{g} \rightsquigarrow \xi_M$ is a v.f. on M defined by $\xi_M|_x := \frac{d}{dt}\Big|_{t=0} (\exp(t\xi).x)$ for $x \in M$.

Hamiltonian group actions

$$\begin{tikzcd} G \circlearrowright (M,\omega) \arrow[r, "\mu", "\text{momentum map}"'] \arrow[d, squiggly] & \mathfrak{g}^* \\ \boxed{\forall a \in G \quad a^*\omega = \omega} & \end{tikzcd}$$

Defn. A group action is **hamiltonian** if $\exists$ a momentum map (often one requires it be equipped with one).

↳ Momentum maps tell us when/whether a group action can be quantized.

Defn (momentum map).

↳ μ is G-equivariant (w.r.t. the coadjoint action). (not everyone assumes this).

↳ $\forall \xi \in \mathfrak{g}$

$$\mu^{\xi} := \langle \mu, \xi \rangle : M \longrightarrow \mathbb{R}$$
$$x \longmapsto \langle \mu(x) \mid \xi \rangle$$

satisfies **Hamilton's equation**:

$$d\mu^{\xi} = \iota(\xi_M)\omega \qquad \text{(up to sign convention)}$$

i.e. $\forall$ v.f. X on M:

$$d\mu^{\xi}(X) = \omega(\xi_M, X).$$

E.g. $\underset{\text{scalar multiplication}}{S^1 \circlearrowright} \mathbb{C}^n = \mathbb{R}^{2n} \xrightarrow{\ \mu\ } \text{Lie}(S^1) \cong \mathbb{R}$

We can take

$$\mu(z_1, \ldots, z_n) \underset{\text{depending on conventions}}{=} -\frac{1}{2}\left(|z_1|^2 + \cdots + |z_n|^2 + \text{const.}\right)$$

For const. $= -1$:

$$\frac{\mu^{-1}(0)}{G} = \frac{S^{2n-1}}{S^1} = \mathbb{P}^{n-1}(\mathbb{C})$$

[Marsden & Weinstein]

Symplectic Reduction

$$M/\!\!/G := \frac{\mu^{-1}(0)}{G}$$

ω descends to a symplectic form $\frac{\mu^{-1}(0)}{G}$ (at least when 0 is a regular value and G acts freely and properly on $\mu^{-1}(0)$).

$$\begin{tikzcd}
\mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d] & M \\
\frac{\mu^{-1}(0)}{G} &
\end{tikzcd}$$

Indeed:

↳ $i^*\omega$ is invariant

↳ $i^*\omega$ is horizontal

$$\iota_{\xi_M}(i^*\omega) = i^*(\iota_M\omega) = i^*(d\mu^\xi)$$
$$= d(\iota^*\mu^\xi) = 0$$

the reduced symplectic form is nondegenerate by dimension counting.

Office Hour

Given $P : G \curvearrowright V$ a group action on a v.s.

we get $\rho : \mathfrak{g} \curvearrowright V$

$P : G \longrightarrow \mathrm{Aut}(V)$ group hom.

$\rho := dP|_e : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ Lie algebra homomorphism.

Special case: $V = \mathfrak{g}$, $P = \mathrm{Ad} \Rightarrow \rho = \mathrm{ad}$

For matrix groups,

$$\mathrm{Ad} : a \longmapsto (B \mapsto aBa^{-1})$$

$$\mathrm{ad} = \dot{\mathrm{Ad}} : A \longmapsto \left(B \mapsto \frac{d}{dt}\Big|_{t=0} (a_t B a_t^{-1})\right)$$

where $\frac{d}{dt} a_t = A$, $a_0 = \mathbb{1}$.

E.g. $a_t = \mathbb{1} + tA$ (may not lie in G, but lives in $\mathrm{Mat}(\mathbb{R})$).

$$\mathrm{ad} : A \longmapsto \left(B \longmapsto \frac{d}{dt}(\mathbb{1} + tA) B (\mathbb{1} + tA)^{-1}\right)$$

$$(\mathbb{1} + tA)^{-1} = \mathbb{1} - tA + t^2A^2 - t^3A^3 + \cdots$$

$$(\mathbb{1} + tA) B (\mathbb{1} + tA)^{-1} = (\mathbb{1} + tA) B (\mathbb{1} - tA + t^2A^2 - t^3A^3 + \cdots)$$

$\frac{1}{1+\varepsilon} = 1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \cdots$

$$= t(AB - BA) + B + \text{h.o.t.}$$

$$\overset{\cancel{\mathrm{ad}}}{\underset{\frac{d}{dt}\big|_{t=0}}{\longmapsto}} AB - BA$$

We can also take $a_t = \exp(tA) = \mathbb{1} + tA + \cdots$

"In Lie theory, it is healthy to treat Lie groups and algebras as matrices"

— Yael

$$H^k(M;\mathbb{Z}) \xrightarrow{\varphi} H^k(M;\mathbb{R}) \cong H^k_{dR}(M)$$

$$H^k_{dR}(M)_{\mathbb{Z}} := \operatorname{im} \varphi$$

$$= \left\{ [\alpha] \;\middle|\; \forall A \in H_k(M), \int_A \alpha \in \mathbb{Z} \right\}$$

A is represented by

$$[A] = \left[\sum_{j=1}^{r} (\sigma_j : \Delta^k \to M) \right]$$

* Not all smooth homology classes are represented by smooth submanifolds.