Recall The big lemma :

$$M := \mathbb{P}^1(\mathbb{C}) \times (\mathbb{R}/\lambda\mathbb{Z})^{2n-2}$$

$$\omega := \omega_{FS} \oplus \omega_{std}$$

$$J_0 := J_{\mathbb{P}^1(\mathbb{C})} \oplus J_{std}$$

$\forall J \in \mathcal{J}(M,\omega)$

$\forall p \in M \ \exists$ J-holomorphic curve $\varphi : \mathbb{P}^1(\mathbb{C}) \to M$ through p with $[\varphi] = [\mathbb{P}^1(\mathbb{C}) \times \{*\}] \in H_2(M)$.

Note: $J_0 \in \mathcal{J}(M,\omega)$.

WLOG assume M is connected.

The big lemma states for $M = \mathbb{P}^1(\mathbb{C}) \times (\mathbb{R}/\lambda\mathbb{Z})^{2n-2}$ and $A = [\mathbb{P}^1(\mathbb{C}) \times \{*\}]$, the map

$$\pi \times ev : \mathcal{M}_A \times \mathbb{P}^1(\mathbb{C}) \twoheadrightarrow \mathcal{J}(M,\omega) \times M$$

is onto.

A foray into point-set topology

Consider $g: A \to B$
- A, B topological spaces,
- g continuous
- B Hausdorff & $\underbrace{\text{compactly generated}}$

$\forall C \subset B$, C is closed $\Leftrightarrow \forall K \subset B$ compact, $K \cap C$ is compact

E.g. metrizable $\Rightarrow$ compactly gen.

loc. compact Hausdorff $\Rightarrow$ compactly gen.

Indeed, $\mathcal{M}_A$, $\mathcal{J}(M,\omega)$ with the C^∞-topology are compactly generated.

Defn. g is proper if the image of the restriction of $g^{-1}: \mathcal{P}(B) \to \mathcal{P}(A)$ to compact sets in B is contained in the compact sets in A.

For B compactly generated:

↳ A compact $\Rightarrow g: A \to B$ proper

↳ $g: A \to B$ proper and $B' \subseteq B$, then
$$g| : g^{-1}(B') \to B'$$
is proper.

↳ $g: A \to B$ proper and $A' \subset A$ closed, $\Rightarrow g|_{A'}$ proper

↳ $g: A \to B$ proper (and B compactly generated) $\Rightarrow g$ is closed.

↳ So, if g is a continuous proper bijection $\Rightarrow g$ is homeo.

↳ If B connected, $g: A \to B$ proper (hence closed), and g open, then g is onto.

We want to apply this to $\pi \times ev : \mathcal{M}_A \times \mathbb{P}^1(\mathbb{C}) \to \mathcal{J}(M,\omega) \times M$:

Let $f : \mathbb{P}^1(\mathbb{C}) \longrightarrow M$ J-holomorphic

$h : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$ holomorphic of degree 1.

$\Rightarrow h \circ f : \mathbb{P}^1(\mathbb{C}) \longrightarrow M$ is J-holomorphic and

$$[h \circ f] = [f]$$

<u>Note</u> : $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$

Fact

$$\{\text{hol. } \mathbb{P}^1(\mathbb{C}) \circlearrowleft \text{ of deg. } 1\}$$

$$= \{\text{Möbius transformations of } \mathbb{P}^1(\mathbb{C})\}$$

$$= \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{C},\ ad-bc \neq 0 \right\}$$

$$\cong GL_2(\mathbb{C}) / \mathbb{C}^*$$

$$= PSL_2(\mathbb{C})$$

$h \in PSL_2(\mathbb{C}) \circlearrowright \mathcal{M}_A \times \mathbb{P}^1(\mathbb{C})$ preserves $\pi \times ev$

$$h.((f,J),z) := ((h^{-1} \circ f, J), h(z))$$

$PSL_2(\mathbb{C}) \circlearrowright \mathcal{M}_A$ preserves π

$h.(f,J) = (h^{-1} \circ f, J)$

"Gromov Compactness"

$$\mathcal{M}_A / PSL_2(\mathbb{C}) \xrightarrow{\pi} \mathcal{J}(M,\omega) \text{ is } \underline{\text{proper}}$$

Easy Corollary!

$$\frac{\mathcal{M}_A \times \mathbb{P}^1(\mathbb{C})}{PSL_2(\mathbb{C})} \xrightarrow{\pi \times ev} \mathcal{J}(M,\omega) \times M \text{ is proper.}$$

So: $\boxed{\text{im}(\pi \times ev) \text{ is closed.}}$

Office Hour

Defn (intersection form) $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$

Volumes of balls via symplectic geometry

(Duistermaat-Heckman)

$$\varphi : \mathbb{R}^2 = \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$$

$$re^{i\theta} \longmapsto \frac{r^2}{2}$$

$$r\,dr \wedge d\theta = d\left(\frac{r^2}{2}\right) \wedge d\theta$$

$$\varphi_* m = 2\pi m$$

standard Lebesgue measure wrt $d\left(\frac{r^2}{2}\right) \wedge d\theta$

More generally,

$$\mathbb{C}^n \longrightarrow \mathbb{R}^n_{\geq 0} \quad \leftarrow \text{positive orthant !}$$

$$(z_i)_i \longmapsto \left(\frac{|z_i|^2}{2}\right)_i$$

$$|z_1|^2 + \cdots + |z_n|^2 \leq 1$$

What is this shape?