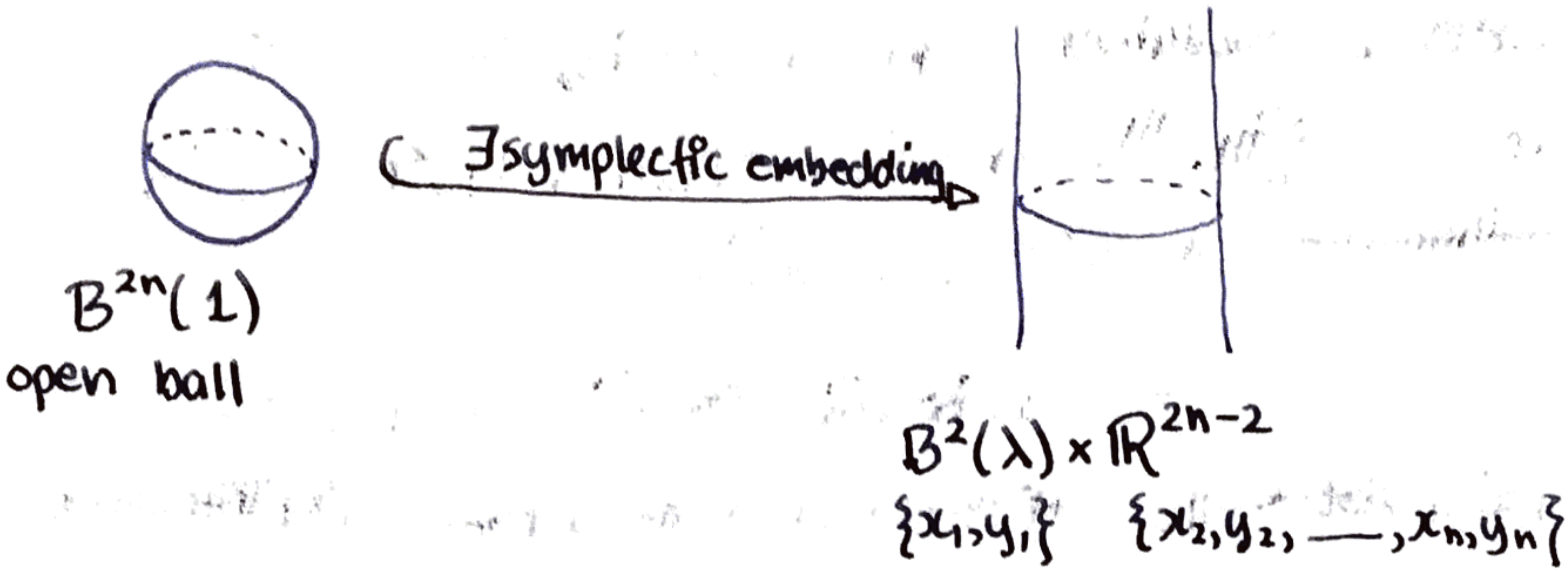
Fact For M^n connected and orientable,

$$\text{dim } H^n_{dR}(M) = \begin{cases} 1, & M \text{ compact} \\ 0, & \text{else.} \end{cases}$$

On Presentations:

- 30-40 minutes
- 2-3 page LaTeX summary (max 5 page)
- Tell Yael your project topic soon

Recall: Gromov nonsqueezing:



THEN $\boxed{\lambda \geq 1}$

Note: 'embedding' $\Leftrightarrow$ diffeomorphism to a subset of the target

> **Fact** Given an injective immersion
> $$f : \underset{\text{open}}{U \subseteq \mathbb{R}^m} \longrightarrow \mathbb{R}^m$$
> Then $f(U) \subseteq \mathbb{R}^m$ is open. Furthermore, f is a diffeomorphism onto its image.

Proof. Since $\dim \operatorname{dom} f = \dim \operatorname{cod} f$, by linear algebra,

f immersion $\Rightarrow$ f submersion.

By the inverse function theorem, f is an open map and f is a local diffeomorphism.

Since f is injective $\exists f^{-1} : \underbrace{f(U)}_{\text{open}} \longrightarrow U$ is smooth.

Remark on PS #1.2: In cylindrical coordinates, (R, θ, z), R denotes the distance to the z-axis, not the distance to the origin. We have the relation $R^2 + z^2 = 1$ in addition to $x = R\cos\theta$, $y = R\sin\theta$

on PS #1.1: Given an isometric embedding ψ of $B^m(1)$ into ~~$\{\vec{x} \mid$ dist $\in$~~ $\mathbb{R}^m$, find an elementary proof that its image is the ball of radius 1 about $\psi(0)$.

$B^m(1) = \{x \mid \mathrm{dist}(x,0) \leq 1\}$

Note: [Brouwer, 1912] "Invariance of domain"

$f : \underset{\text{open}}{U \subseteq \mathbb{R}^m} \longrightarrow \mathbb{R}^m$ injective & continuous, then f is a homeo. onto and open subset.

</digressions>

For Gromov nonsqueezing, it is enough to prove:

(∗) If $\exists \epsilon > 0$ such that $B^{2n}(r+\epsilon) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2}$ is a symplectic embedding, then $r \leq R$

Proof (of sufficiency). Suppose (∗) and take $B^{2n}(1) \hookrightarrow B^2(\lambda) \times \mathbb{R}^{2n-2}$ a symp. emb. Let $\epsilon > 0$ By (∗) with $\cancel{R}\, r = 1 - \epsilon$, we obtain $1 - \epsilon \leq \lambda$ for all $\epsilon > 0$; thus $\lambda \geq 1$.

Proof (of (*)). Assume $B^{2n}(r+\epsilon) \overset{\varphi}{\hookrightarrow} B^2(R) \times \mathbb{R}^{2n-2}$ is a symp. embedding. Consider

$$\operatorname{image}\left(\overline{B^{2n}(r+\tfrac{\epsilon}{2})} \xrightarrow{\varphi|} B^2(R)\times\mathbb{R}^{2n-2} \xrightarrow{pr_{\mathbb{R}^{2n-2}}} \mathbb{R}^{2n-2}\right) =: C$$

C is compact, thus bounded. Choose $\lambda > \operatorname{diam}(C)$.

We get

$$\begin{tikzcd}
B^{2n}(R+\frac{\epsilon}{2}) \arrow[r, hook] \arrow[dr, "\psi"'] & B^2(R) \times (\mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2}) \arrow[d, "\substack{\text{add a point}\\ \text{to } B^2(\mathbb{R})}"] \\
& \mathbb{P}^1(\mathbb{C}) \times \dfrac{\mathbb{R}^{2n-2}}{\lambda\mathbb{Z}^{2n-2}} =: M
\end{tikzcd}$$

$$R^2\omega_{FS} \oplus \omega_{std} =: \omega$$

Claim $\exists J \in \mathcal{J}(M,\omega)$ such that

$$\psi|_{B^{2n}(r)} : (B^{2n}(r), J_{std}) \to (M, J)$$

is holomorphic. (as almost complex manifolds).

Proof. $J_1 := \psi_* J_{std}$ is a compatible complex structure on $\psi(B^{2n}(r+\frac{\epsilon}{2}))$. By flexibility of almost complex structures, $\exists J \in \mathcal{J}(M,\omega)$ s.t. $J|_{\psi(B^{2n}(r+\frac{\epsilon}{2}))} = J_1$.

Pf... We have

$$\psi : (B^{2n}(r), \omega_{std}, J_{std}) \hookrightarrow \Big(\mathbb{P}^1(\mathbb{C}) \times \frac{\mathbb{R}^{2n-2}}{\lambda\mathbb{Z}^{2n-2}}, \underbrace{R^2\omega_{FS} \oplus \omega_{std}}_{=:\,\omega}, J\Big)$$

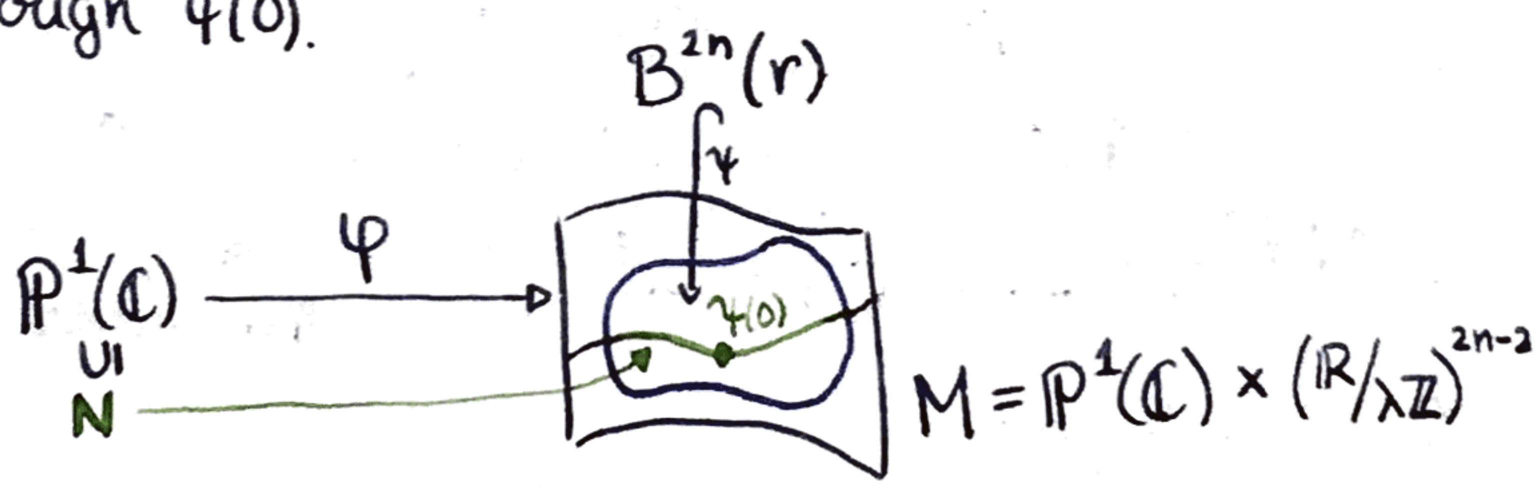
We know $J, J_0 := J_{\mathbb{P}^1(\mathbb{C})} \oplus J_{\mathbb{C}^{n-1}} \in \mathcal{J}(M,\omega)$, but

$\psi_* J_{std} = J$

Big Lemma $\forall p \in M$ $\exists J$-holomorphic curve through p $\varphi : \mathbb{P}^1(\mathbb{C}) \to M$ such that $[\varphi] = [\mathbb{P}^1(\mathbb{C}) \times \{*\}]$

Notice: $\forall p \in M$, $\exists J_0$-holomorphic curve $\varphi : \mathbb{P}^1(\mathbb{C}) \to M$ through p such that $[\varphi_0] = [\mathbb{P}^1(\mathbb{C}) \times \{*\}]$. Namely, if $p = (z_0, v_0)$ for $z_0 \in \mathbb{P}^1(\mathbb{C})$ and $v_0 \in \mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2}$, take $\boxed{\varphi_0(z) = (z, v_0).}$

Fix $\varphi : \mathbb{P}^1(\mathbb{C}) \to M$ J-holomorphic in $[\mathbb{P}^1(\mathbb{C}) \times \{*\}]$ through $\psi(0)$.



Let $N := \varphi^{-1}(\psi(B^{2n}(r))) \underset{open}{\subseteq} \mathbb{P}^1(\mathbb{C})$

Proof... φ holomorphic $\overset{\text{details}}{\underset{\text{later}}{\Longrightarrow}}$ $\varphi^*\omega$ is an <u>area form</u>, so

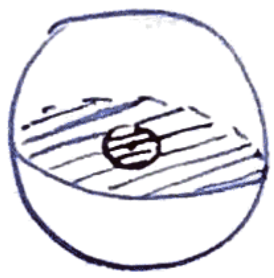
$$\int_N \varphi^*\omega \leq \int_{\mathbb{P}^1(\mathbb{C})} \varphi^*\omega$$

$$\Psi^{-1}\circ\varphi| : N \subseteq \mathbb{P}^1(\mathbb{C}) \longrightarrow (B^{2n}(r), J_{std})$$

is a holomorphic curve. Furthermore, it is <u>proper</u>: Preimages of compact sets in $B^{2n}(r)$ are compact in N (details in HW or later).

By a <u>consequence</u> of <u>Wirtinger's Equality</u> → details to follow

$$\underset{\substack{\text{Riemann}\\\text{surface}}}{N} \xrightarrow[\substack{\text{holomorphic,}\\\text{proper,}\\\text{through } 0.}]{f} B^{2n}(r), \text{ then } \int_N f^*\omega_{std} \geq \pi r^2$$

E.g.  → $N \neq$ as f wouldn't be proper.

So,

$$\pi r^2 \underset{\text{Wirtinger}}{\leq} \int_N \underbrace{(\Psi^{-1}\circ\varphi)^*\omega_{std}}_{\varphi^*\circ\underbrace{(\Psi^{-1})^*\omega_{std}}_{\omega}} = \int_N \varphi^*\omega \underset{\varphi \text{ holomorphic}}{\leq} \int_{\mathbb{P}^1(\mathbb{C})} \varphi^*\omega$$

$$= \int_{\underset{[\mathbb{P}^1(\mathbb{C})\times\{*\}]}{\overset{\shortparallel}{[\varphi]}}} \omega = \int_{\mathbb{P}^1(\mathbb{C})\times\{*\}} \omega = \int_{\mathbb{P}^1(\mathbb{C})} R^2\omega_{FS} = R^2\pi$$

$\blacksquare$

Let us now rephrase the Big Lemma:

Let (M,ω) be compact and symplectic; let $A \in H^2(M)$.

Consider the...

Universal Moduli Space

$$\mathcal{M}_A := \left\{ (f,J) \;\middle|\; \begin{array}{l} J \in \mathcal{J}(M,\omega) \\ f: \mathbb{P}^1(\mathbb{C}) \to M \ \ J\text{-hol., } [f]=A \end{array} \right\}$$

$$\begin{array}{ccc} (f,J) & & \mathcal{M}_A \\ \downarrow & & \downarrow \pi \\ J & & \mathcal{J}(M,\omega) \end{array}$$

We have an evaluation map:

$$\begin{aligned} ev: \mathcal{M}_A \times \mathbb{P}^1(\mathbb{C}) &\longrightarrow M \\ ((f,J),z) &\longmapsto f(z) \end{aligned}$$

$$\begin{aligned} \mathcal{M}_A \times \mathbb{P}^1(\mathbb{C}) &\xrightarrow{\pi \times ev} \mathcal{J}(M,\omega) \times M \\ ((f,J),z) &\longmapsto (J, f(z)) \end{aligned}$$

The "big lemma" states for $M = \mathbb{P}^1(\mathbb{C}) \times (\mathbb{R}/\lambda\mathbb{Z})^{2n-2}$ and $\omega = R^2\omega_{FS} \oplus \omega_{std}$, the map $\pi \times ev$ is onto.