Indecomposable homology classes in a symplectic manifold

"The terminology here is not uniform, so don't take it as coming from God."
— Yael

Defn. Let M be a connected manifold. A homology class $A \in H_2(M)$ is *spherical* if it is in the image of the Hurewicz homomorphism

$$\pi_2(M) \longrightarrow H_2(M)$$

i.e. $\exists \varphi : S^2 \to M$ such that $A = [\varphi] \; (:= \varphi_*[S^2])$.

Defn. A symplectic manifold (M, ω) is *symplectically aspherical* if $[\omega] \in H^2_{dR}(M)$ vanishes on spherical homology classes. I.e. $\forall$ smooth $\varphi : S^2 \to M$, $\int_{S^2} \varphi^* \omega = 0$.

Recall: $H^2_{dR}(M) \cong H^2(M; \mathbb{R}) = H^2(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$
$\cong \mathrm{Hom}(H_2(M), \mathbb{R})$

E.g. $\pi_2 = 0 \Rightarrow$ symplectic asphericality.
↳ For instance, any genus $g \geq 1$ surface Σ_g with an area form.

Eg....

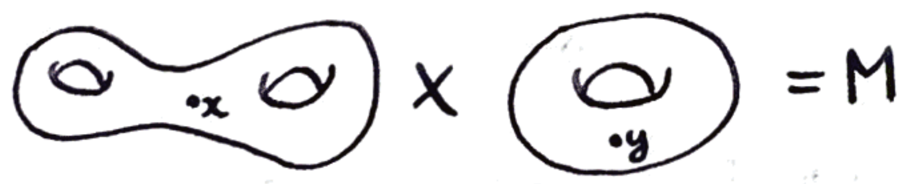
↳ $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, \sum_{j=1}^{n} dx_j \wedge dy_j)$

↳ $(T^*S^2, \omega_{canonical})$ ← this is exact

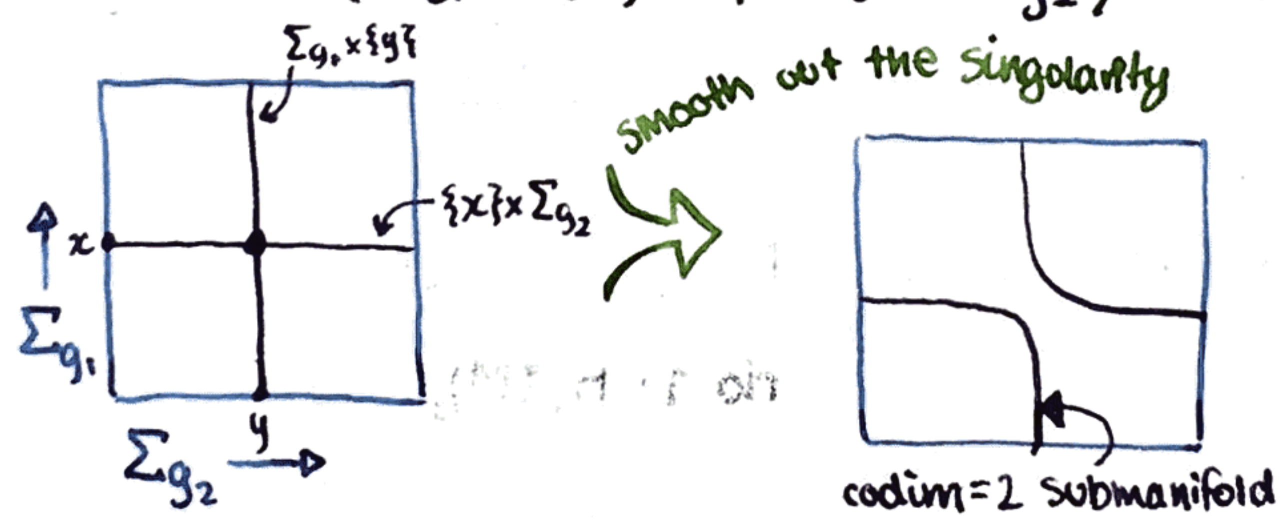
↳ Indeed, any exact symplectic manifold.

In 1998, Robert Gompf constructed a closed symplectic manifold with $\pi_2 \neq 0$ which is symplectically aspherical:

Sketch: Take $g_1, g_2 \geq 1$ and consider $\Sigma_{g_1} \times \Sigma_{g_2} =: M$



Consider $(\Sigma_{g_1} \times \{y\}) \cup (\{x\} \times \Sigma_{g_2}) \subseteq M$



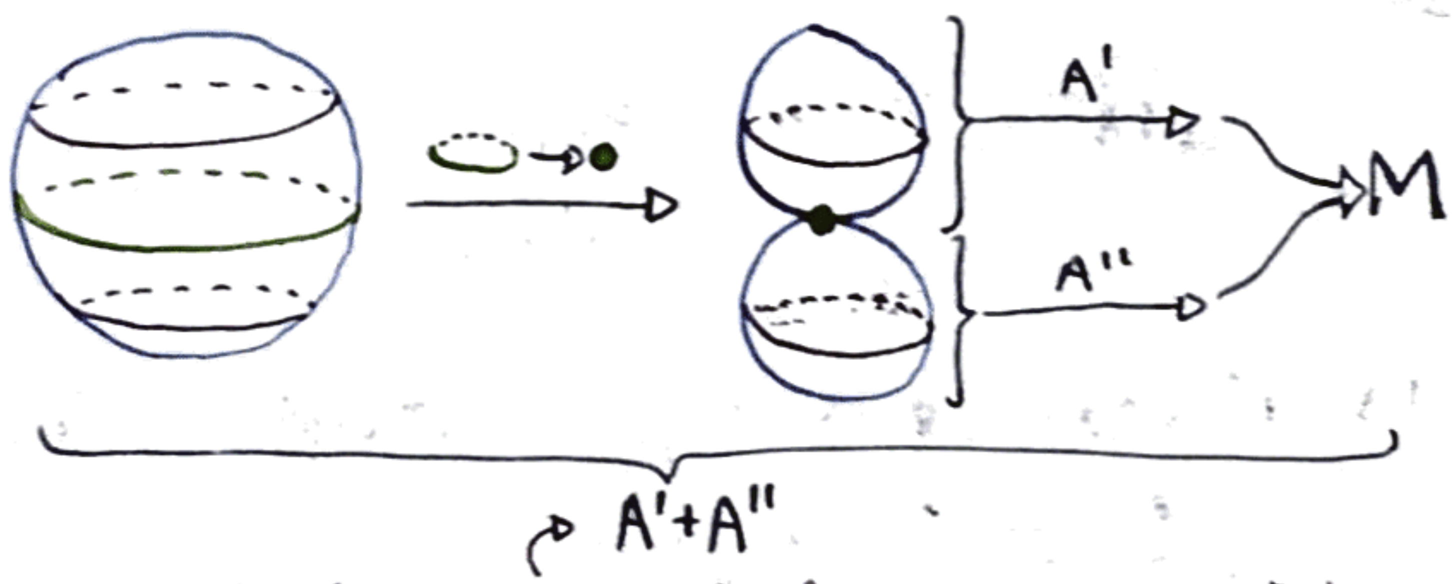
codim=2 allows for a notion of a branched covering. e.g. $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ branched @ 0.

$\exists$ symplectic analogue. Gompf constructed a symplectic branched covering of $\Sigma_{g_1} \times \Sigma_{g_2}$ along this codim-2 symplectic submanifold.

↪ The notion of a symplectic branched cover is due to Gromov, which holds for any symplectic submanifold $N \subset (M,\omega)$ st. $\operatorname{codim} N = 2$.

Defn. Given (M,ω) symplectic and $A \in H_2(M)$ spherical with $\omega(A) > 0$, A is indecomposable if there existsn't a decomposition $A = A' + A''$ into $A', A'' \in H_2(M)$ spherical with $\omega(A'), \omega(A'') > 0$.

Recall: Composition in $\pi_2(M)$ is given by collapsing the equator:



homotope the maps until they share a basepoint

E.g.

Assume $\omega(\pi_2(M)) \subseteq \mathbb{R}$ is a discrete subgroup of $\mathbb{R}$. (e.g. if $[\omega] \in H^2_{dR}(M)_{\mathbb{Z}}$) and is generated by $\omega(A)$. Then A is indecomposable.

Proof. Otherwise, $A = A' + A''$, for A', A'' spherical classes with positive symplectic area. Then

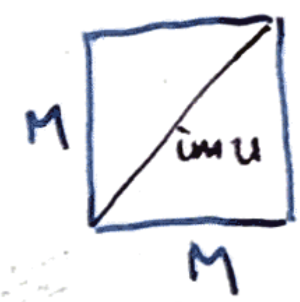
$$\omega(A) = \underbrace{\omega(A')}_{} + \underbrace{\omega(A'')}_{}$$

positive elements of $\omega(\pi_2(M)) \subseteq \mathbb{R}$

Thus $\omega(A') = k'\omega(A)$, $\omega(A'') = k''\omega(A)$ for some $k', k'' \in \mathbb{Z}_{\geq 1}$. So $\omega(A) = (k' + k'')\omega(A)$ and $k' + k'' > 1$. $\lightning$

Interesting nonexample: Let $(M, \omega) = (\mathbb{P}^1(\mathbb{C})^{\times 2}, \omega_{FS}^{\oplus 2})$

$$\begin{aligned} u : \mathbb{P}^1(\mathbb{C}) \cong S^2 &\longrightarrow M \\ z &\longmapsto (z, z) \end{aligned}$$



Then: $[u] = [\mathbb{P}^1(\mathbb{C}) \times \{*\}] + [\{*\} \times \mathbb{P}^1(\mathbb{C})] \in H_2(M)$

So $[u]$ is not indecomposable.

E.g. $(M, \omega) = (\mathbb{P}^1(\mathbb{C}) \times V, \omega_{FS} \oplus \omega_V)$ where (V, ω_V) is a symplectically aspherical manifold. Then

$$A := [\mathbb{P}^1(\mathbb{C}) \times \{*\}]$$

is symplectically aspherical.

Office Hour

↳ The symplectic form ω on a vector space V is exact, so the cohomology class $[\omega]=0$.

↳ $\sum dx_j \wedge dy_j = d(\sum x_j dy_j)$

In polar coordinates

$$\sum r_j dr_j \wedge d\theta_j = \sum ds_j \wedge d\theta_j = d(\sum s_j d\theta_j) \quad (*)$$

$$s_j := \frac{r_j^2}{2}, \quad ds_j = r_j dr_j$$

For $\theta = \arctan\frac{y}{x}$,

$$d\theta = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{xdy-ydx}{x^2} = \frac{xdy-ydx}{x^2+y^2}$$

So $s d\theta = \frac{r^2}{2} d\theta = \frac{1}{2}(xdy-ydx)$

$$(*) = d\left(\sum \tfrac{1}{2}(x_j dy_j - y_j dx_j)\right)$$

Given a differential form α of compact support st. $\int_M \alpha = 0$, then α is exact. This is why

$$H_c^{top}(M) \cong \mathbb{R}$$

compact support

for even noncompact manifolds.

Fact For M^n connected and orientable,

$$\text{dim} \, H^n_{dR}(M) = \begin{cases} 1, & M \text{ compact} \\ 0, & \text{else.} \end{cases}$$