Review of Homology

Defn. <u>Singular k-chains in M</u> are:

$$\sum m_\sigma \sigma \in \mathbb{Z}\{\Delta^k \to M \text{ cts.}\}$$

That is, formal integer combinations of continuous maps from the k-simplex to M.

We have a boundary map:

$$\partial : \left\{ \begin{matrix} \text{singular} \\ k\text{-chains} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{singular} \\ (k-1)\text{-chains} \end{matrix} \right\}$$

$$H_k(M) := H_k(M, \mathbb{Z}) := \frac{\{k\text{-cycles}\}}{\{k\text{-boundaries}\}} := \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}$$

singular homology

Defn. <u>Smooth singular homology</u> is the same as above, but with "continuous" replaced with "smooth".

Fact The natural map

$$H_k^{\text{smooth}}(M) \xrightarrow{\sim} H_k(M)$$

is an isomorphism.

Functoriality: H_k is a (covariant) functor:

$f: M \to M'$ induces $f_* := H_k(f) : H_k(M) \to H_k(M')$

↳ $H_k(g \circ f) = H_k(g) \circ H_k(f)$

↳ $H_k(\mathrm{Id}_M) = \mathrm{Id}_{H_k(M)}$

↳ If f is a diffeomorphism, $H_k(f)$ is an isomorphism.

↳ If M^n is closed, oriented, and connected, then $H_n(M) \cong \mathbb{Z}$, generated by the fundamental class, $[M]$. (The orientation determines which of ± 1 generate $\mathbb{Z}$).

↳ If $M^n \overset{i}{\hookrightarrow} W$ is an oriented embedded submanifold, then we have $[i] := i_*[M] \in H_n(W)$

Moreover, given any map $f: M^n \to W$ from an oriented n-mfld to W. Denote $[f] := f_*[M] \in H_n(W)$

Cohomology

Let R be an abelian group (e.g. $\mathbb{Z}, \mathbb{R}, \mathbb{Z}/2\mathbb{Z}$).

Defn. A singular k-cochain with coefficients in R is a homomorphism $\underbrace{\mathbb{Z}\{\Delta^k \xrightarrow{\text{cts}} M\}}_{\text{singular } k\text{-chains}} \longrightarrow R$

Defn. The coboundary map $\delta : \left\{ \begin{matrix} \text{singular} \\ k \text{ cochains} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{singular} \\ (k+1) \text{ cochains} \end{matrix} \right\}$ defined via $\delta_k := \partial^*_{k+1}$

We then get the cohomology groups

$$H^k(M;R) := \frac{\{k\text{-cocycles}\}}{\{k\text{-coboundaries}\}} = \frac{\ker \delta_k}{\operatorname{im} \delta_{k-1}}$$

We can do integration! Given $\alpha \in \Omega^k(M)$ and A a smooth k-cycle

$\int_A \alpha \in \mathbb{R}$. If $A = \sum m_j \sigma_j$, $\sigma_j : \Delta^k \to M$

$$\int_A \alpha := \sum_j m_j \int_{\Delta^k} \sigma_j^* \alpha$$

↳ If $f: M \to M'$ is smooth, then $\int_{f_* A} \alpha = \int_A f^* \alpha$

↳ If $f: M \to M'$ is a diffeomorphism of connected, oriented manifolds and α is of top degree on M', then

$$\int_M f^* \alpha = \pm \int_{M'} \alpha$$

depending whether f preserves or reverses orientation

The map $A \longmapsto \int_A \alpha$ is a homomorphism from $\mathbb{Z}[\{\Delta^k \xrightarrow{\text{cts}} M\}]$ to $\mathbb{R}$, i.e an $\mathbb{R}$-coefficiented cochain

Varying α gives us $\Omega^k(M) \longrightarrow \{k \text{ cochains}\}$. In cohomology, this induces

$$H^k_{dR}(M) \xrightarrow{\sim} H^k(M;\mathbb{R}) \cong \mathrm{Hom}(H_k(M),\mathbb{R})$$

de Rahm cohomology,
or differential Real coh.

Defn. For M, M' connected, closed, oriented manifolds of the same dimension and $\Psi: M \to M'$ is smooth, the **degree** of Ψ is the $d \in \mathbb{Z}$ such that

$$[\Psi] = \Psi_*[M] = d \cdot [M']$$

If Ψ is a diffeomorphism then $d = \pm 1$.

Defn. $H^k_{dR}(M)_{\mathbb{Z}} := \mathrm{image}\left(H^k(M;\mathbb{Z}) \to H^k(M;\mathbb{R}) \cong H^k_{dR}(M)\right)$

called **integral de Rahm cohomology**.

$$= \left\{ [\alpha] \in H^k_{dR}(M) \;\middle|\; \forall \text{smooth cycles } A, \int_A \alpha \in \mathbb{Z} \right\}$$

This gives us a lattice inside $H^k_{dR}(M)$.

Product Structure

↳ Suppose now the abelian group R also has a Ring structure (e.g. $\mathbb{Z}, \mathbb{R}, \mathbb{Z}/2\mathbb{Z}$).

Defn. The cup product $H^k(M) \otimes_R H^\ell(M) \longrightarrow H^{k+\ell}(M)$

$$a \otimes b \longmapsto a \smile b$$

In de Rahm cohomology, $[a] \smile [b] := [a \wedge b]$ for a, b closed differential forms.

Recalso: For M^n closed, connected, oriented,

$$H^n_{dR}(M) \cong \mathbb{R} \text{ via } [\alpha] \mapsto \int_M \alpha$$

Corollaries for symplectic manifolds

↳ Let (M^{2n}, ω) be compact and symplectic.

Then for any symplectic structure ω_0 on $\mathbb{R}^{2N}$, there does not exist a symplectic embedding

$$(M, \omega) \hookrightarrow (\mathbb{R}^{2N}, \omega_0)$$

Proof. ω^n is a volume form and M is compact, so $\int_M \omega^n > 0$

So $[\omega]^n = [\omega^n] \neq 0 \in H^{2n}_{dR}(M)$, so $[\omega] \neq 0 \in H^2_{dR}(M)$.

But, for any symplectic form ω_0 on $\mathbb{R}^{2N}$, $[\omega_0] = 0$.

E.g. The nonzero homology groups of $\mathbb{P}^n(\mathbb{C})$ are

$$H_{2k}(\mathbb{P}^n(\mathbb{C})) = \mathbb{Z}[\mathbb{P}^k(\mathbb{C})]$$

for $k=0,1,\_\_,n$; where $\mathbb{P}^k(\mathbb{C}) \hookrightarrow \mathbb{P}^n(\mathbb{C})$ is induced from $\mathbb{C}^{k+1} \hookrightarrow \mathbb{C}^{n+1}$

The nonzero integral de Rahm cohomology of $\mathbb{P}^n(\mathbb{C})$ are

$$H^{2k}_{dR}(\mathbb{P}^n(\mathbb{C}))_{\mathbb{Z}} = \langle [\omega^k] \rangle$$

for $k=0,1,\_\_,n$ and $\omega = \frac{1}{\pi}\omega_{FS}$

Corollary If $(M,\omega) \hookrightarrow (\mathbb{P}^n(\mathbb{C}),\omega_{FS})$ then $\frac{1}{\pi}[\omega] \in H^2_{dR}(M)_{\mathbb{Z}}$.

Thus, let $M := S^2 \times S^2$, $\omega_{a,b} := a\omega_{S^2} \oplus b\omega_{S^2}$ for $a \geq b > 0$

standard area form

If $\frac{a}{b} \notin \mathbb{Q}$, then $\forall r \in \mathbb{R}$, $\nexists\, \iota : (S^2 \times S^2, \omega_{a,b}) \hookrightarrow (\mathbb{P}^n(\mathbb{C}), r\omega_{FS})$

Exercise 6.1 If n is even, there does not exist an orientation reversing diffeomorphism of $\mathbb{P}^n(\mathbb{C})$.

Exercise 6.2 If $(S^2\times S^2, \omega_{a,b}) \cong (S^2\times S^2, \omega_{a',b'})$ are symplectomorphic and $a\geq b>0$, $a'\geq b'>0$, then $a=a'$ and $b=b'$.

Exercise Assuming Gromov's nonsqueezing theorem, prove: for $D^2(r)$ the disc of radius r, If $(D^2(a)\times D^2(b), \omega_{std}) \cong (D^2(a')\times D^2(b'), \omega_{std})$ and $a\geq b>0$, $a'\geq b'>0$, then $a=a'$, $b=b'$. ($\omega_{std} = dx_1\wedge dy_1 + dx_2\wedge dy_2$).

More corollaries of $[\omega]^n \neq 0$:

- The only even sphere which admits a symplectic form is S^2 (If $2n>2$, then $H^2(S^{2n})=\{0\}$).
- The product $S^2\times S^4$ does not admit a symplectic form ($H^2(S^2\times S^4) = \langle [\omega_{S^2}]\oplus 0 =: a\rangle$. $a^2=0 \Rightarrow a^3=0$).

"All symplectic manifolds admit a compatible almost complex structure, but if you see one on the street or in a coffee shop, it usually also has a complex structure."

— Yael

Let (M,ω) be a compact Kähler manifold.

$\dim_{\mathbb{C}} M = n$, $\dim_{\mathbb{R}} M = 2n$

Hard Lefschetz ~~Theorem~~ Property $\forall k$ s.t. $0 \leq k \leq n$

$$\begin{aligned} H^k_{dR}(M) &\xrightarrow{\sim} H^{2n-k}_{dR}(M) \\ a &\longmapsto a \smile [\omega]^{n-k} \end{aligned}$$

is an isomorphism.

Not all compact symplectic manifolds satisfy this hard Lefschetz property. However, we do not know a compact 1-connected (M,ω) with a circle $S^1 \circlearrowright M$ with $\#\{\text{fixed points}\} < \infty$ which does not satisfy hard Lefschetz