Intro to Symplectic Geometry

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During the course, feel free to review topics in the prereq list (manifolds, homology, flows, etc.). There are various crash-courses available on the course website.

Participation:

- Interrupt Yael during class; it is okay to say things wrong.

Work:

- Homework assignments
- Oral presentations

Topics:

- Gromov nonsqueezing theorem
- Classical mechanics is geometrized by symplectic geometry.

Yael's interests:

- Hamiltonian group actions
- Geometric quantization

Symplectic forms & Volume

- Riemannian structures allow you to perform measurements on a manifold; symplectic structures allow you to measure other things.
- Volume is a coarser notion of measurement.

Defn. A symplectic manifold is a pair (M, ω) with M a smooth (i.e. C^∞) manifold, and ω a closed, non-degenerate 2-form.

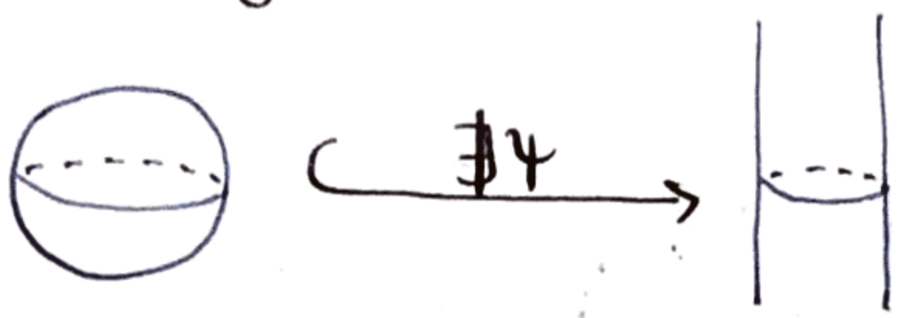
$\hookrightarrow d\omega = 0$ $\quad$ $\hookrightarrow$ "every (nonzero) vector has a friend" $\quad$ $\hookrightarrow$ antisymmetric!

E.g. $\left(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i\right)$, with coords $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Darboux Theorem All symplectic manifolds look like the above example locally. I.e. $\exists$ coords $\underbrace{x_1, \ldots, x_n, y_1, \ldots, y_n}$ s.t. $\omega = \sum dx_i \wedge dy_i$.

These are called Darboux, symplectic, or canonical coordinates.

Gromov nonsqueezing theorem In $\mathbb{R}^{2n}$, consider the cylinder $\{x_i^2 + y_i^2 < \lambda\} \subset \mathbb{R}^{2n}$. Then, if $\lambda < 1$, there exists no symplectic embedding of the open unit ball in $\mathbb{R}^{2n}$ into the cylinder.



Exercise $\nexists$ isometric embedding of $B \hookrightarrow$ the cylinder

Defn. $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ is symplectic or a symplectomorphism if it is a diffeomorphism and $\psi^* \omega_2 = \omega_1$.

Symplectic Linear Algebra

Lemma (Symplectic Gram-Schmidt)

Let V be a vector space and $\Omega : V \otimes V \to \mathbb{R}$ a nondegenerate $\underbrace{\text{antisymmetric bilinear form}}_{\text{2-covector}}$ (Ω is called a symplectic tensor by John Lee).
Then $\exists$ basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ such that $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$.

such a basis is called standard

Corollary $\dim V \in 2\mathbb{Z}$.

Remark: In this basis, $\Omega(v, w) = v^T \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix} w$.

Proof. If $V = \{0\}$, we are done.

Otherwise, since Ω is nondegenerate, $\exists e_1, f_1$ such that $\Omega(e_1, f_1) = 1$. Ω antisymmetric means $\dim \underbrace{\operatorname{span}\{e_1, f_1\}}_{W} = 2$.

Pf... Let $W^{\Omega} = \{u \in V \mid \forall w \in W \ \Omega(u,w) = 0\}$

$$= \{u \in V \mid \Omega(u, e_1) = \Omega(u, f_1) = 0\}$$

called the symplectic (ortho)complement of W.

Claim: $V = W \oplus W^{\Omega}$

Proof. • $W \cap W^{\Omega} = \{0\}$, since W is symplectic, every nonzero vector has a friend.

• $W + W^{\Omega} = V$, since $\forall v \in V$,

$$v = \Omega(v, f_1) e_1 + \Omega(e_1, v) f_1 + w$$

for some $w \in W^{\Omega}$ (solve $v - \lambda e_1 - \mu f_1 \in W^{\Omega}$ for $\lambda, \mu \in \mathbb{R}$).

Exercise: Ω is nondegenerate on W^{Ω}

By induction, $\exists$ a standard basis for W^{Ω}, say $e_2, \ldots, e_n, f_2, \ldots, f_n$. It follows that

$$\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$$

is a standard basis for V. ∎

Exercise: Let Ω be an antisymmetric bilinear form on a vector space V. The nullspace

$$\mathrm{Null}(\Omega) := \{u \in V \mid \Omega(u,\cdot) \equiv 0\}$$

measures the failure of nondegenericity. Consider $\pi : V \longrightarrow \bar{V} = V/\mathrm{Null}\,\Omega$. Then

- $\exists!$ antisymmetric bilinear form $\bar{\Omega}$ on $\bar{V}$ such that $\Omega = \pi^*\bar{\Omega}$ (i.e. $\forall u, v \in V$, $\Omega(u,v) = \bar{\Omega}(\pi u, \pi v)$)
- $\bar{\Omega}$ is nondegenerate.

Lemma For V an $\mathbb{R}$-v.s. and Ω an antisymmetric bilinear form, TFAE:

1. Ω is nondegenerate
2. $\mathrm{Null}\,\Omega = \{0\}$
3. $\Omega^{\#} : V \xrightarrow{\sim} V^*$ is an isomorphism.
   $v \longmapsto \Omega(v,\cdot) =: \underset{\substack{\uparrow \\ \text{contraction / pullback}}}{v \lrcorner \Omega} \doteq \iota(v)\Omega \doteq \iota_v \Omega$
4. $\dim V = 2n$ is even and $\underset{\text{"}\Omega^{\wedge n}\text{"}}{\Omega^n} \neq 0$.

Proof. ($1 \Rightarrow 4$) Suppose Ω is nondegenerate. Then in a standard basis $e_1, \ldots, e_n, f_1, \ldots, f_n$,

Pf...
$$\Omega = e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*$$

where $\{e_1^*, \_\_, e_n^*, f_1^*, \_\_, f_n^*\}$ is the dual basis. It follows

$$\Omega^{\wedge n} = n!\, e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* \neq 0$$

($4 \Rightarrow 1$) Suppose Ω is degenerate. Then $\bar{\Omega}$ on $\bar{V} = V/\mathrm{Null}\,\Omega$ is nondegenerate. since $\dim \bar{V} < 2n$, and $\deg \bar{\Omega}^{\wedge n} = 2n > \dim \bar{V}$, $\bar{\Omega}^{\wedge n} = 0$, so $\Omega^{\wedge n} = \pi^*(\bar{\Omega}^{\wedge n}) = (\pi^* \bar{\Omega})^{\wedge n} = 0^{\wedge n} = 0$. ■

Corollary For (M, ω) a symplectic manifold of dimension $2n$, there exists the Liouville volume form $\frac{1}{n!}\omega^n$. This induces both an orientation and a measure on M!

In symplectic coordinates, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

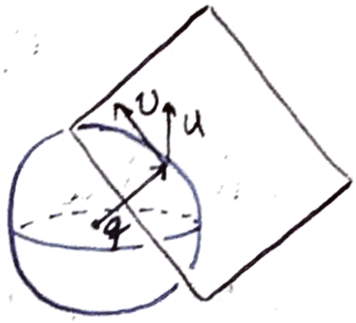
$$\frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

In dim = 2, a symplectic form is an area form!

Exercise:

Consider the following area form on S^2:

$$\omega_q(u,v) = \langle q, u \times v \rangle$$



for $q \in S^2$, $u, v \in T_q S^2 = q^\perp$

Show:

(i) ω is invariant under $SO(3)$. Without introducing coordinates, state and prove this.

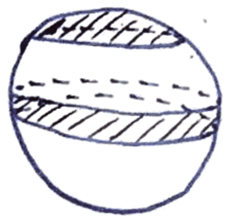
(You may use that $SO(3)$ is orthogonal and $SO(3) \circlearrowright (\mathbb{R}^3, \times)$ as a Lie algebra).

(ii) In cartesian coordinates,

$$\boxed{\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}$$

(iii) In cylindrical coordinates outside the poles; R, θ, z, $x = R\cos\theta$; $y = R\sin\theta$. Then $\boxed{\omega = d\theta \wedge dz}$

This theorem is due to Archimedes!



E.g. (Symplectomorphism) For $0<\lambda<1$

$$\psi : (\mathbb{R}^{2n}, \omega) \longrightarrow (\mathbb{R}^{2n}, \omega)$$
$$(x_1, y_1, \ldots, x_n, y_n) \mapsto (\lambda x_1, \tfrac{1}{\lambda} y_1, \ldots, \lambda x_n, \tfrac{1}{\lambda} y_n)$$

This gives an embedding of the unit ball into the cylinder $\{x_1^2 + x_2^2 + \cdots + x_n^2 < \lambda^2\}$.

In Gromov nonsqueezing, the cylinder is a nbhd of $\mathbb{R}^{2n-2}_{x_2, y_2, \ldots, x_n, y_n}$, a symplectic subspace.	In the above, the cylinder is a nbhd of $\mathbb{R}^n_{y_1, \ldots, y_n}$, on which ω vanishes.

Defn (orthocomplement) Let (V, Ω) a symplectic VS, and $S \subset V$ a linear subspace. Then

$$S^{\Omega} := \{u \in V \mid \iota_u \Omega|_S = 0\}$$
$$= \{u \in V \mid \forall v \in S, \ \Omega(u, v) = 0\}.$$

E.g. $V = \mathbb{R}^{2n}$, $e_1, \ldots, e_n, f_1, \ldots, f_n$ standard basis, $\Omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*$. Then

- $\mathrm{span}\{f_1, \ldots, f_n\}^{\Omega} = \mathrm{span}\{f_1, \ldots, f_n\}$
- $\mathrm{span}\{e_1, f_1\}^{\Omega} = \mathrm{span}\{e_2, f_2, \ldots, e_n, f_n\}$

Defn. $S \subset V$ is symplectic if $\Omega|_{S\otimes S}$ is nondegenerate.

$\Leftrightarrow V = S \oplus S^{\Omega}$ (ex.)

$S \subset V$ is isotropic if $\Omega|_{S\otimes S} = 0$

$\Leftrightarrow S \subset S^{\Omega}$ (ex.)