p-adic reading group

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October 19, 2018

1 Review of Non-archimedean number fields

We begin with an introduction of notation:

- Henceforth F will denote a non-archimedean local field (of characteristic 0). Think $F = \mathbb{Q}_p$ or $\mathbb{F}_p$.
- $|\cdot|$ is the normalized absolute value, i.e. $d_F(ax) = |a|\, d_F\, x$ for $a \in F^\times$.
- $|\cdot|\colon F^\times \to \mathbb{R}_{>0}$ has image a discrete subgroup $q^{\mathbb{Z}} = \left\{ \dots, q^{-2}, q^{-1}, 1, q, q^2, \dots \right\}$. And $q = p^f (= N_p)$

Recall. F is locally compact, Hausdorff, separable and metrizable and totally disconnected (the only connected subspaces are points).

$\mathcal{O}_F = \left\{ x \in F \;\middle|\; |x| \le 1 \right\}$ is the ring of integers.

- $\mathcal{O}_F$ is compact open.
- F is the field of fractions of $\mathcal{O}_F$.
- $\mathcal{O}_F$ is a PID that hasa unique prime = maximal ideal: $\mathfrak{p} = \left\{ x \in F \;\middle|\; |x| < 1 \right\}$.
- $\mathfrak{p}$ is also compact open.
- $\mathcal{O}_F^\times = \mathcal{O}_F \setminus \mathfrak{p} = \left\{ x \in F \;\middle|\; |x| = 1 \right\}$ is compact open.
- $\mathfrak{p} = (\varpi)$, generated by the/a "uniformizer".
- The ideals of $\mathcal{O}_F$ are:

$$\cdots \subseteq \varpi^2 \mathcal{O}_F = \mathfrak{p}^2 \subseteq \varpi \mathcal{O}_F = p \subseteq \varpi^{-1} \mathcal{O}_F = \mathfrak{p}^{-1} \subseteq \dots \tag{1}$$

- $F = \bigcup_{k \in \mathbb{Z}} \varpi^k \mathcal{O}_F$.
- The field $\mathfrak{k} = \mathcal{O}_F/\mathfrak{p}$ is a finite field of order $\int |\varpi| = q$.
- If $S \subseteq \mathcal{O}_F$ is a fixed set of representations of $\mathfrak{k}$ with $0 \in S$ and if ϖ is a fixed uniformizer, then every $0 \neq x \in F$ can be written uniquely as

$$x = \sum [n \geq N] a_n \varpi^n \tag{2}$$

  for some $N \in \mathbb{Z}$, $a_n \in S$, and $a_N \neq 0$ ($|x| = q^{-N}$).

- $$\mathcal{O}_F = \varprojlim_{k \geq 1} \mathcal{O}_F/\mathfrak{p}^k \tag{3}$$

  which is a limit of finite rings in the category of topological rings.

Let $\underline{G}$ be a connected reductive group over F (e.g. $\underline{G} = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SP}_n$). Then $\underline{G}(F)$ (e.g. $\mathrm{GL}_n(F)$) can be given a topology from F: Choose an embedding $\underline{G} \hookrightarrow \mathrm{GL}_n$, then

$$G := \underline{G}(F) \leq GL_n(F) \subset F^{n^2} \tag{4}$$

Give the subspace topology, we get a separable metric space.

Then $G = \underline{G}(F)$ becomes a separable metric space which is a locally compact Hausdorff group *and* totally disconnected (a td-space).

Lemma 1. *If $\mathbb{P}^F(d)$ is compact, then for every porjective variety $\underline{X}$, $X = \underline{X}(F)$ is compact.*

Definition. A td-space (or locally profinite space) is a topological space X such that one of the following equivalent conditions holds:

1. X is locally compact and totally disconnected.
2. X is Hausdorff and every point $x \in X$ has a neighbourhood basis of compact open sets.

Let us explore some basic properties of td-spaces.

- Every closed (resp. open) subspace of a td space is td.
- Every finite or restricted product of td spaces is td. A restricted product is on with $\prod_{i \in I}(X_i, K_i)$ with almost all the K_i compact opens.
- If X is a first-countable td-space, then every point has a countable neighbourhood basis of compact opens.

- If X is a second-countable td-space, then X has a countable neighbourhood basis of compact opens. (Hint: prove that if $\mathcal{B}$ is a basis, then $\mathcal{B}_{\mathrm{co}} = \{ U \in \mathcal{B} \mid U \text{ compact open} \}$ is a basis).
- If X is a td-space and countable at ∞ (is a countable union of compact sets), then it is a countable increasing union of compact opens.

Proposition 2. *A space X is a compact td-space if and only if it is a profinite space. A profinite space is a projective limit of finite discrete spaces in the category of topological spaces: $X = \varprojlim X_i$.*

Proof. $\implies$): Use Tychinoff's theorem.

$\impliedby$):Since X is compact, every partition $\mathcal{U} = \{U_\alpha\}_\alpha$ of X into opens is finite (U_α is necessarily compact open). Order according to refinement: $\mathcal{U}' \leq \mathcal{U} \iff \mathcal{U}$ is a refinement of $\mathcal{U}'$. From this we get a directed set. We then get a map

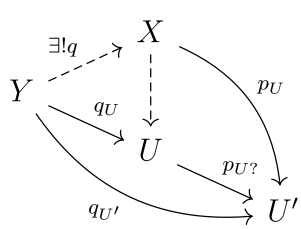
$$\begin{aligned} p_{\mathcal{U},\mathcal{U}'} \colon \mathcal{U} &\to \mathcal{U}' \\ U_\alpha &\mapsto U'_\alpha \supseteq U_\alpha \end{aligned} \tag{5}$$

Claim 3.

$$X = \varprojlim_{\mathcal{U}} \mathcal{U} \tag{6}$$

Observe $p_{\mathcal{U}} \colon X \to \mathcal{U}$ with $x \mapsto U_\alpha \ni x$.

Let Y be a topological space and $\{q_U \colon Y \to \mathcal{U}\}_{\mathcal{U}}$ a compatible family.



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For $y \in Y$, consider $\bigcap_U q_U(y) = \{q(y)\} \subset X$ $\square$

2 Smooth functions on td-spaces

Let K be a compact td space. We want to define $C^\infty(K) = C^\infty_c(K)$.

Note $p_U\colon K \to U$, $p_{U,U'}\colon U \to U'$ are all surjective. View the finite sets U as 0-dimensional manifolds. Then

$$C^\infty(U) = \{f\colon U \to \mathbb{C}\} = C^U \cong C^{\#U} \tag{8}$$

We get injections $C^\infty(U) \hookrightarrow C^\infty(U)$ and $C^\infty(U) \hookrightarrow \mathbb{C}^K$. Our idea is that $\bigcup_U C^\infty(U) \subseteq \mathbb{C}^K$ should be $C^\infty(K)$.
Definition.

$$C^\infty(K) \coloneqq \{\, f\colon K \to \mathbb{C} \mid f \text{ is locally constant} \,\} = \varinjlim_U C^\infty(U) \tag{9}$$

where the limit is in the category of vector spaces.

What topology do we put on $C^\infty(K)$? Perhaps unintuitively, we give it the discrete topology. We also give $\mathbb{C}$ the discrete topology. Why don't we use the injective limit topology? Consider the following analogy:

To justify this, consider that there are a few theories for automorphic forms:

- The K-finite theory, which is far more algebraic, and
- "Smooth", introduced by Cogdel.
- L^2 theory, which is far more analytic.

We will later develop $C_c^\infty(X)$, which will have the discrete topology, and $C^\infty(X)$ where X is not compact, and this space will *not* get the discrete topology.

Let now X be an arbitrary td-space. Smooth functions on X *should* restrict to smooth functions on every K compact open, so smooth should, at the very least, be locally constant. Then,
Definition. For X not compact,

$$C^\infty(X) \coloneqq \{\, f\colon X \to \mathbb{C} \mid f \text{ is locally constant} \,\} \tag{10}$$

$$C_c^\infty(X) \coloneqq \left\{\, f \in C^\infty(X) \;\middle|\; \operatorname{supp} f \text{ is compact} \,\right\} \tag{11}$$

As vector spaces,

$$C^\infty(X) = \varprojlim_{K \text{ c.o.}} C^\infty(K) \tag{12}$$

$$C^\infty(X) = \varinjlim_{K \text{ c.o.}} C^\infty(K) \tag{13}$$

If $K_1 \subset K_2$, the maps are

$$\begin{aligned} C^\infty(K_1) &\to C^\infty(K_2) \\ f &\mapsto f\mathbb{1}_{K_1} \\ f|_{K_1} &\leftarrow\!\dashv f \end{aligned} \tag{14}$$

We also have $C^\infty(X) \twoheadrightarrow C^\infty(K)$ by restriction and $C^\infty(K) \to C^\infty(X)$ by extension by 0. The topology on $C^\infty_{\mathrm{c}}(X)$ is the finest one such that $C^\infty(K) \to C^\infty_{\mathrm{c}}(X)$ are all continuous, which is the discrete topology.

The topology on $C^\infty(X)$ is the coarsest such that the above holds. Convergence is given by a net $\langle f_\alpha \rangle_\alpha$ in $C^\infty(X)$ converges to $f \in C^\infty(X)$ if and only if for all X, $f_\alpha|_K \to f|_K$. This is equivalent to $f\alpha|_K$ is eventually $f|_K$ for all K, known as the "compact convergent topology" (with respect to the discrete topology on $C^\infty(K)$.

3 Distributions

We want to take continuous duals of these spaces to define distributions.

$$\mathcal{D}(X) \coloneqq C^\infty_{\mathrm{c}}(X)' = \left\{\xi \colon C^\infty_{\mathrm{c}}(X) \to \mathbb{C} \text{ linear}\right\} \tag{15}$$

$$\mathcal{D}_{\mathrm{c}}(X) \coloneqq C^\infty(X)' = \left\{\xi \colon C^\infty_{\mathrm{c}}(X) \to \mathbb{C} \text{ linear, cts wrt cpt. conv. top, disc. top. on } \mathbb{C}\right\} \tag{16}$$

$$\tag{17}$$

In both cases, we get the weak* topology with respect to the discrete topology on $\mathbb{C}$.

3.1 Td-groups

Recall $G = \underline{G}(F)$ is a topological groups (separable, metrizable) and a td-space.

Definition. A td-group is a topological group that is a td-space.

Lemma 4 (von Danttzig)**.** *Let G be a topological group. Then the following are equivalent:*

1. G *is a td-group.*

2. $1 \in G$ *has a neighbourhood basis of compact open subgroups.*

3. $1 \in G$ *has a neighbourhood basis of profinite (open?) subgroups.*

Lemma 5. *Compact, td-groups are exactly the profinite groups (projective limits of finite groups in* **TopGrp**.

Recommendation: Fiona's notes on the homspace are easier to understand.

4 Ch. 3: Haar Measures and Hecke Algebras

Definition. If G is a td space, the representation $\pi\colon G \to \mathrm{GL}(V)$ is smooth if the map $g \mapsto \pi(g)v$ is locally constant.

The space of smooth reps can be viewed as $\mathcal{D}_{\mathrm{c}}(G)$-modules.

Let (π, V) be a smooth rep of G. Let $\xi \in \mathcal{D}_{\mathrm{c}}(G)$, $v \in V$. We want to define $\pi(\xi)v$, giving us the action of $\mathcal{D}_{\mathrm{c}}(G) \curvearrowright V$. Let $K \subset G$ be compact open, $K \supseteq \xi$. Since $g \mapsto \pi(g)v$ is a locally constant map, restriction to K yields:

$$\pi(g)v = \sum_{i=1}^{n} \mathbb{1}_{K_i}(g)v_i \tag{18}$$

where $K_1, \ldots, K_n$ are open compact subsets of K, $v_1, \ldots, v_n \in V$. This maps goes from $K \to V$. If $g \in K_i$, $\pi(g)v = v_i$. We set $\pi(\xi)v = \sum_{i=1}^{n} \xi(\mathbb{1}_{K_i})v_i$. This gives V a $\mathcal{D}_{\mathrm{c}}(G)$-module structure.

Remark. All smooth functions of compact support are a finite linear combination of indicator functions over compact open subsets.

Remark (A note on convolution)**.** The multiplication $\mu\colon G \times G \to G$ induces a comultiplication

$$\begin{aligned} \mu^*\colon C^\infty(G) &\to C^\infty(G \times G) \\ f &\mapsto f \circ \mu \end{aligned} \tag{19}$$

Then if ξ, $\xi' \in \mathcal{D}_{\mathrm{c}}(G)$, we define their convolution

$$(\xi * \xi')(f) = (\xi \boxtimes \xi')(\mu^* f) \tag{20}$$

Note this is not necessarily commutative.

From the $\mathcal{D}_{\mathrm{c}}(G)$-module structure, we can recover the action $G \curvearrowright V$ by setting $\pi(g)v \coloneqq \pi(\delta_g)v$.

Let G be a td group. Then $G \times G \curvearrowright G$ via left, right translation:

$$l_x y \coloneqq xy \qquad\qquad r_x y \coloneqq yx^{-1} \tag{21}$$

with induced actions on $C_{\mathrm{c}}^{\infty}(G)$ being

$$l_x \phi(y) = x^{-1} y \qquad\qquad r_x \phi(y) = \phi(yx) \tag{22}$$

We define $l_x \xi$, $r_x \xi$ for $\xi \in \mathcal{D}(G)$ by

$$\langle l_x \xi, \phi \rangle = \langle \xi, l_{x^{-1}} \phi \rangle \qquad\qquad \langle r_x \xi, \phi \rangle = \langle \xi, r_{x^{-1}} \phi \rangle \tag{23}$$

For instance, if $\xi = \delta_y$, we have $l_x \delta_y = \delta_{xy}$ and $r_x \delta_y = \delta_{yx^{-1}}$.

Proposition 6. *The space $\mathcal{D}_c(G)^{l(G)}$ of left-invariant distributions is a* 1*-dimensional* $\mathbb{C}$*-vector space.*

Proof. Let K_i, $i \in I$ denote the system of neighbourhoods of the identity element, consisting of compact open subgroups. Let $C_{\mathrm{c}}^{\infty}(G/K_i) \coloneqq \left\{ f \in C_{\mathrm{c}}^{\infty}(G) \;\middle|\; \forall x \in K_i, r_x f = f \right\}$. If $K_j \subset K_i$, then $C_{\mathrm{c}}^{\infty}(G/K_i) \hookrightarrow C_{\mathrm{c}}^{\infty}(G/K_j)$, and thus the spaces $C_{\mathrm{c}}^{\infty}(G/K_i)$, together with inclusion maps, form a direct system. Then

$$\varinjlim_{K_i} C_{\mathrm{c}}^{\infty}(G/K_i) = C_{\mathrm{c}}^{\infty}(G) \tag{24}$$

as every $f \in C_{\mathrm{c}}^{\infty}(G)$ is right-invariant under some K_i, for sufficiently small K_i. Thus, $\mathcal{D}(G) = \varprojlim_{K_i} \mathcal{D}(G/K_i)$ and $\mathcal{D}(G)^{l(G)} = \varprojlim_{K_i} \mathcal{D}(G/K_i)^{l(G)}$. For all K_i, $C_{\mathrm{c}}^{\infty}(G/K_i)$ has basis $\mathbb{1}_{xK_i}$, where xK_i are left cosets. A distribution $\xi \in \mathcal{D}(G/K_i)$ is G-invariant (under l_g) if and only if $\xi(\mathbb{1}_{xK_i}) = \xi(\mathbb{1}_{K_i})$ for all $x \in G$. In other words,

$$\begin{aligned} \mathcal{D}(G/K_i)^{l(G)} &\to \mathbb{C} \\ \xi &\mapsto \xi(\mathbb{1}_{K_i}) \end{aligned} \tag{25}$$

is an isomorphism. This proves $\mathcal{D}(G/K_i)^{l(G)}$ is 1-dimensional. If $K_i \supset K_j$, we have $C_{\mathrm{c}}^{\infty}(G/K_i) \subset C_{\mathrm{c}}^{\infty}(G/K_j)$, and we have a surjection $\mathcal{D}(G/K_j) \twoheadrightarrow \mathcal{D}(G/K_i)$, and a map between 1-dimensional spaces $\mathcal{D}(G/K_j)^{l(G)} \to \mathcal{D}(G/K_i)^{l(G)}$. Let $\xi_i \in \mathcal{D}(G/K_j)^{l(G)}$, ξ_i, its image in $\mathcal{D}(G/K_i)^{l(G)}$. Then ξ_i, ξ_j are completely determined by $\xi(\mathbb{1}_{K_j})$ and $\xi(\mathbb{1}_{K_i})$. Since $K_i = \bigsqcup_{x \in K_i/K_j} xK_j$, we have $\xi(\mathbb{1}_{K_i}) = \#(K_i/K_j)\xi(\mathbb{1}_{K_j})$. Hence elements of $\mathcal{D}(G)^{l(G)}$ consist of systems of $\alpha_i \in \mathbb{C}$ such that $\alpha_i = \#(K_i/K_j)\alpha_i$ which forms a $\mathbb{C}$-vector space. □

Proposition 7. *Let K be a compact td space. Then left-invariant distributions on K are right-invariant.*

4.1 Modulus character and unimodular groups

Since $C_c^\infty(G) \curvearrowright G$, by left, right translation and these commute with each other $(r_x(l_y f) = l_y(r_x f))$, the space $\mathcal{D}(G)^{l(G)}$ is stable under right-translation.

Since $\mathcal{D}(G)^{l(G)}$ is 1-dimensional, there exists a unique group homomorphism $\Delta_G \colon G \to C^\times$ such that for all $\mu \in \mathcal{D}(G)^{l(G)}$, we have $r_G\mu = \Delta_G(g)\mu$ for all $g \in G$. Note that if for every compact open $K \subset G$, $Delta_G|_K = 1$, then Δ_G is a smooth character.
Definition. A td space is unimodular if $\Delta_G = 1$.
Proposition 8.

1. *For any smooth character $\chi \colon G \to \mathbb{C}^\times$, the space $\mathcal{D}(G)^{l(G,\chi)}$ of distributions μ such that $l_g\mu = \chi(g)\mu$ for all $g \in G$, is 1-dimensional. Morever, the map $\mu \mapsto \chi^{-1}\mu$ defines an isomorphism of $\mathbb{C}$-vector spaces $\mathcal{D}(G)^{l(G)} \to \mathcal{D}(G)^{l(G,\chi)}$.*
2. *$\mathcal{D}(G)^{r(G,\chi)}$ is 1-dimensional and the map $\mu \mapsto \chi_\mu$ gives an isomorphism $\mathcal{D}(G)^{r(G)} \to \mathcal{D}(G)^{r(G,\chi)}$.*
3. *We have $\mathcal{D}(G)^{r(G,\chi)} = \mathcal{D}(G)^{r(G,\Delta_G\chi)}$. In particular, $\mathcal{D}(G)^{l(G)} = \mathcal{D}(G)^{r(G,\Delta_G)}$ and $\mathcal{D}(G)^{r(G)} = \mathcal{D}(G)^{l(G,\Delta_G)}$.*
4. *If $\mu \in \mathcal{D}(G)^{l(G)}$, then $\Delta_G\mu$ is a right-invariant distribution.*

Proof. 1. It is enough to show for all χ, χ' of G smooth characters, the multiplication $\mu \mapsto \chi\mu$ defines a map $\mathcal{D}(G)^{l(G,\chi')} \longleftrightarrow \mathcal{D}(G)^{l(G,\chi^{-1}\chi')}$. Then $\mu' \mapsto \chi\mu'$ would give its inverse. In particular, this holds for $\chi' = 1$.

We need to check for $\mu \in \mathcal{D}(G)^{l(G,\chi')}$, then $\chi_\mu \in \mathcal{D}(G)^{l(G,\chi^{-1}\chi')}$,

$$l_g\chi\mu = \chi(g)^{-1}\chi l_g\mu \tag{26}$$

on $\mathcal{D}(G)$. Then

$$l_g\mu = \chi'(g)\mu = \chi(g)^{-1}\chi l_g\mu = (\chi^{-1}\chi')(g)(\chi\mu) \tag{27}$$

So $\chi\mu \in (\mathcal{D}(G)^{l(G,\chi^{-1}\chi')})$. Then $\mu \mapsto \chi\mu$ such that $\langle \chi\mu, \phi\rangle = \langle \mu, \chi\phi\rangle$. For all $\phi \in C_c^\infty(G)$,

$$\left\langle \chi(g)^{-1}\chi l_g\mu, \phi\right\rangle = \chi(g)^{-1}\left\langle l_g\mu, \chi\phi\right\rangle = \chi(g)^{-1}\left\langle \mu, l_{g^{-1}}\chi\phi\right\rangle \tag{28}$$

$$\left\langle l_g\chi\mu, \phi\right\rangle = \left\langle \chi\mu, l_{g^{-1}}\phi\right\rangle = \left\langle \mu, \chi l_{g^{-1}}\phi\right\rangle \tag{29}$$

Thus equation (26) follows from: $\chi(g)^{-1}l_{g^{-1}}(\chi\phi) = \chi l_{g^{-1}}\phi$ (exercise).

2. Similar
3. Follows from the previous two. □

Example. On $G = F \rtimes F^\times$, $r_{-x} = (x_1, t_1) = (x_1 + t_1 x, t_1)$, and $\Delta_G(x,t) = q^{\mathrm{ord}(t)} = |t|_f^{-1}$

4.2 The inverse involution

The map $g \mapsto g^{-1}$ of $G \to G$ induces an involution on $C^\infty(G)$, $C_c^\infty(G)$, $\mathcal{D}(G)$, $\mathcal{D}_c(G)$. For distrubutions, we will denote this by $\xi \mapsto \check{\xi}$, called the inverse involution.

$$\langle \check{\xi}, f\rangle = \langle \xi, \check{f}\rangle \tag{30}$$

where $\check{f}(g) = f(g^{-1})$. We have the formula $(\xi_1 * \xi_2)^\vee = \check{\xi}_2 * \check{\xi}_1$. The inverse involution acts as an adjoint operation $\forall \xi \in \mathcal{D}_c(G)$, $\xi_1 \in \mathcal{D}_c(G)$, and $\phi_1 \in C^\infty(G)$,

$$\langle \xi * \xi_1, \phi_1\rangle = \langle \xi_1, \check{\xi} * \phi_1\rangle \tag{31}$$

Proposition 9. *$\xi \mapsto \check{\xi}$ defines an isomorphism $\mathcal{D}(G)^{l(G)} \to \mathcal{D}(G)^{r(G)}$. More precisely, for all $\mu \in \mathcal{D}(G)^{l(G)}$, $\check{\mu} = \Delta_G^{-1}\mu$.*

Proof. $\check{\mu}$, $\Delta_G^{-1}\mu$ are both vectors in the 1-dimensional space $\mathcal{D}(G)^{r(G)}$. It is enough to find $\phi \in C_c^\infty(G)$ sich that $\langle \check{\mu}, \phi\rangle = \langle \Delta_G^{-1}\mu, \phi\rangle \neq 0 \in \mathbb{C}$. Take $\phi = \mathbb{1}_K$ for any $K \leq G$ compact open. □

4.3 Hecke Algebras

Definition. The Hecke algebra of a td group G is

$$\mathcal{H}(G) = \left\{\, \xi \in \mathcal{D}_c(G) \;\middle|\; \xi \text{ smooth wrt } l_g \,\right\} \tag{32}$$

For all compact open $K \leq G$, let $e_K = \mathbb{1}_K \mu(K)^{-1}\mu$, where $\mu \in \mathcal{D}(K)^{l(K)}$ and $\mu(K)^{-1} = \mu(\mathbb{1}_K)^{-1}$. e_K is independend of all choices. We have $\delta_g * e_K = e_K * \delta_g = e_K$ for all $g \in K$, and $e_K * e_K = e_K$, so e_K is idempotent. Observe $\check{e}_K = e_K$ (*Hint:* show this first for $\phi = \mathbb{1}_{K'}$).

Proposition 10. *A vector $\xi \in \mathcal{D}_c(G)$ is smooth with respect to l_g if and only if there exists a compact open $K \leq G$ where $e_K * \xi = \xi$.*

Proof. If $l_g\xi = \xi$ for all $g \in X$, then we show $e_K * \xi = \xi$ by definition:

$$\begin{aligned} \langle e_K * \xi, f\rangle &= \int_G \int_G f(xy)\, de_K(x)\, d\xi(y) \\ &= \int_G \int_G f(xy)\, d\xi(y)\, de_K(x) \\ &= \int_G f(xy)\, d\xi(y) = \langle \xi, f\rangle \end{aligned} \tag{33}$$

"I just Fubinied it there" – Hannah

If $_X * \xi = \xi$, then for all $g \in K$, we have $l_g\xi = \delta_g * \xi = \delta_g * (e_K * \xi) = e_K * \xi = \xi$ by assumption. ☐

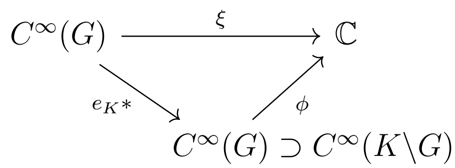
Proposition 11.

1. *$\xi \in \mathcal{D}_c(G)$ smooth with respect to l_G if and only if $\xi = \phi\mu$, where $\phi \in C_c^\infty(G)$ and $\mu \in \mathcal{D}_c * G)^{r(G)}$.*
2. *ξ is smooth with respect to l_g if and only if it is smooth with respect to r_g. In other words, $\xi \in \mathcal{H}(G) \iff \exists K \leq G, e_K * \xi = \xi * e_K = \xi$.*
3. *If $\xi_1 \in \mathcal{H}(G)$, $\xi_2 \in \mathcal{D}_c(G)$, then $\xi_1 * \xi_2 \in \mathcal{H}(G)$, and $\xi_2 * \xi_1 \in \mathcal{H}(G)$. This implies $\mathcal{H}(G)$ is a two-sided ideal, hence $\mathcal{H}(G)$ is a subalgebra of $\mathcal{D}_c(G)$.*

Proof.

1. If $\xi \in \mathcal{H}(G)$, then there exists $K \leq G$ such that for all $g \in K$, $l_g\xi = \xi \iff e_K * \xi = \xi$. If $C := supp\xi$, then $C \subseteq G$ is compact open and invariant under left translation of K. For all $\psi \in C^\infty(G)$, $\langle \xi, \psi\rangle = \langle e_K * \xi, \psi\rangle = \langle \xi, \check{e}_K * \psi\rangle$. The image of $\psi \mapsto e_K * \psi$ is the subspace of $C^\infty(G)$ consisting of left K-invariant functions on G. We

can identify this with $C^\infty(K\backslash G)$



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A continuous linear form on $C^\infty(K\backslash G)$ is give by $\phi\colon K\backslash G \to \mathbb{C}$ with finite support. We can identify ϕ with a function with compact support in G. Then $\xi = \phi\mu$, for some $\mu \in \mathcal{D}(G)^{r(G)}$ such that $\mu(\mathbb{1}_K) = 1$.

2. If μ is a right-invariant distribution on G, $\Delta_G\mu$ is left-invariant on G. Hence $\xi \in \mathcal{D}_\mathrm{c}(G)$ is smooth with respect to l_g, hence with respect to r_g. Since there exists K such that $r_G\xi = \xi$ for all $g \in K$, then $\Delta_K\xi$ is left-invariant on K, so $l_g\xi = \xi$ for all $g \in K$.

3. If $\xi_1 \in \mathcal{H}(G)$, then $\exists K \leq G$ such that $E_K * \xi_1 = \xi_1 * e_K = \xi_1$. Then $\xi_1 * \xi_2 = e_K * \xi_1 * \xi_2$, and $\xi_2 * \xi_1 = \xi_2 * \xi_1 * e_K$, so $\xi_1 * \xi_2 \in \mathcal{H}(G)$ and $\xi_2 * \xi_1 \in \mathcal{H}(G)$. □

Corollary 12. *If G is a unimodular td group and μ a non-zero Haar distribution on G, then $\phi \mapsto \phi\mu$ induces a $G \times G$-equivariant isomorphism $\mu\colon C_c^\infty(G) \to \mathcal{H}(G)$.*

Proof. To check equivariance, use $l_{g_1}r_{g_2}(\phi\mu) = \big(l_{g_1}r_{g_2}(\phi)\big)\big(l_{g_1}r_{g_2}(\mu)\big) = \mu$. □

Observe $\mathcal{H}(G)$ is in general non-unital, since δ_{e_G} of $\mathcal{D}_\mathrm{c}(G)$ is not a smooth distribution unless G is discrete. But $\mathcal{H}(G)$ has many idempotents (including the e_K) which "replace" the identity.

For any compact open $K \leq G$, we consider the subalgebra $\mathcal{H}_K(G) \coloneqq e_K * \mathcal{H}(G) * e_K$. Then e_K is the unit of $\mathcal{H}_K(G)$. We have $\mathcal{H}(G) = \bigcup_K \mathcal{H}_K(G)$. This leads to the notion of an idempotented algebra: in general, for A an associative algebra with $E(A)$ its set of idempotents, then there is a partial ordering on $E(A)$ given by

$$e \leq f \iff ef = fe = e \tag{35}$$

If $e \in E(A)$, then eAe is a unital subalgebra with unit e. We say A is an idempotented algebra if $A = \bigcup_{e\in E(A)} eAe$.

In $\mathcal{H}(G)$, we have $e_K \leq e_{K'} \iff K \supset K'$. Moreover, every idempotent $e \in \mathcal{H}(G)$ is dominated by some e_K. Hence $\mathcal{H}(G)$ is idempotented.

4.4 Non-degenerate modules over $\mathcal{H}(G)$

Let (π, V) be a smooth representation of G. The action $G \curvearrowright V$ can be extended to an action of $\mathcal{D}_c(G)$ via

$$\pi(\xi)v = \sum_{i=1}^{n} \xi(\mathbb{1}_{K_i})v_i \tag{36}$$

By restricting to $\mathcal{H}(G)$, we see V has the structure of and $\mathcal{H}(G)$-module. The smoothness of V as a representation of G can be translated as a property of the corresponding $\mathcal{H}(G)$-module.

Proposition 13. *Let (π, V) be a representation of G such that $\mathcal{D}_c(G) \curvearrowright V$ by extending $G \curvearrowright V$. For all $\mathbb{K} \leq G$, $v \in V^K$ (the K-invariants) if and only if $v = \pi(e_K)v$. Thus $V^K = \pi(e_K)V$.*

Proof. If $\pi(g)v = v$ for all $g \in K$, then $\pi(e_K)v = v$ by definition.

Conversely, if $\pi(e_K)v = v$,then for all $g \in K$,

$$\pi(g)v = \pi(g)\pi(e_K)v = \pi(\delta_g)\pi(e_K)v = \pi(\delta_g * e_K)v = \pi(e_K)v = v \tag{37}$$

Moreover, if $v \in \pi(e_K)V$, then $\pi(e_K)v = v$ because e_K is idempotent. □

Definition. Let A be an idempotented algebra. We say a A-module M is non-degenerate if $M = \bigcup_{e \in E(A)} eM$. Note: for all $e \in E(A)$, eM is an eAe-module.

For $\mathcal{H}(G)$, M is nondegenerate if and only if $M = \bigcup_K e_K M$ where the union is over all compact open $K \leq G$, since every $e \in E(\mathcal{H}(G))$ has $e \leq e_K$ for some K.

Proposition 14. *There is an equivalence of categories between smooth representations of G and non-degenerate $\mathcal{H}(G)$-modules.*

Proof. If v is a smooth representation of G, then for all $v \in V$, the map $g \mapsto \pi(g)(v)$ is locally constant, thus for all $K \leq G$ such that $\pi(G)v = v$ for all $g \in K$. It follows that $V = \bigcup_K \pi(e_K)V$ where K ranges over the compact open subgroups of G. It follows V is non-degenerate. □

If V is non-degenerate, i.e.

$$V = \bigcup_{e \in E(\mathcal{H})} \pi(e)V \tag{38}$$

we may extend the action of $\mathcal{H}(G) \curvearrowright V$ to an action of $\mathcal{D}_{\mathrm{c}}(G)$. For all $\xi \in \mathcal{D}_{\mathrm{c}}(G)$ and $v \in V$, we choose $K \leq G$ such that $\pi(e_K)v = v$. Then $\pi(\xi)v = \pi(\xi(e_K)v$, where $\xi * e_K \in \mathcal{H}(G)$.

Claim 15. *This definition is independent of the choice of e_K. Indeed, if $e \leq e'$, then $e' * e = e$ and we have $\pi(\xi * e')v = \pi(\xi * e')\pi(e)v = \pi(\xi * e)v$ for all $v \in e_K V \subset e_{K'} V$.*

For all $g \in G$, we set $\pi(g)v = \pi(\delta_g)v$. Since $\delta_{gg'} = \delta_g \delta_{g'}$, this yields a group homomorphism $G \to \mathrm{GL}(V)$.

Claim 16. *For all $v \in V$, $g \mapsto \pi(g)v$ is smooth. Let $K \leq G$ with $\pi(e_K)v = v$. Then $\pi(g)v = \pi(\delta_g)v = \pi(\delta_g * e_K)v$, and so $g \mapsto \pi(g)v$ is right-K invariant, hence smooth.*

Definition. Let (π, V) be a representation of G. A vector $v \in V$ is called smooth if $g \mapsto \pi(g)v$ is smooth. That is, if there exists $K \leq G$ such that $\pi(g)v = v$ for all $g \in K$.

The space of smooth vectors in V is denoted $V^{\mathrm{sm}} = \bigcup_K V^K$. Assume that $G \curvearrowright V$ can be extended as $\mathcal{D}_{\mathrm{c}}(G) \curvearrowright V$, then for all $K \leq G$, $\pi(e_K)V = V^K$, $V^{\mathrm{sm}} = \bigcup_K \pi(e_K)V$.

5 Contragredient and admissible representations

Let (π, V) be a smooth representation of G. Let V^* denote the space of all linear functionals $v^* \colon V \to \mathbb{C}$. Then we have an action $G \curvearrowright V^*$ by $v^* \mapsto \pi^*(g)v^*$, satisfying

$$\left\langle \pi^*(g)v^*, v \right\rangle = \left\langle v^*, \pi(g^{-1})v \right\rangle \tag{39}$$

V^* is also a $\mathcal{D}_{\mathrm{c}}(G)$-module via $\left\langle \pi^*(\xi)v^*, v \right\rangle = \left\langle v^*, \pi(\check{\xi})v \right\rangle$.

Definition. In general, (π^*, V^*) is not smooth. We define the contragredient of V as

$$V' = \left\{ v' \in V^* \;\middle|\; \exists K \leq G, \pi^*(e_K)v' = v' \right\} \tag{40}$$

That is, the collection of smooth vectors of V^*.

Let V be a $\mathbb{C}$-vector space and V^* its dual. Then there exists a natural associative algebra structure on $V \otimes V^*$ given by

$$(v_1 \otimes v_1^*)(v_2 \otimes v_2^*) = \langle v_1^*, v_2 \rangle (v_1 \otimes v_2^*) \tag{41}$$

Define $\mathrm{End}_{\mathrm{fin}}(V) \subset \mathrm{End}(V)$ those linear transformations of V with finite image.

Proposition 17. *Let V be a $\mathbb{C}$-vector space. By assigning to each vector $w = \sum_{i=1}^n v_i \otimes v_i^*$ the map $f_w(v) = \sum_{i=1}^n v_i \langle v_i^*, v \rangle$, we get $V \otimes V^* \xrightarrow{\sim} \mathrm{End}_{fin}(V)$. Additionally, we get $G \times G$-equivariant algebra isomorphism $V \otimes V' \to \mathrm{End}_{fin}(V)^{sm}$.*

Definition. A smooth rep (π, V) is <u>admissible</u> if for all $K \leq G$, $V^K = \pi(e_K)V$ is finite dimensional.

Proposition 18. *If (π, V) is an admissible representation, then the operator $\pi(\phi)$ has finite dimensional image (where $\phi \in \mathcal{H}(G)$).*

This implies for an admissible representation (π, V), the algebra $\pi \colon \mathcal{H}(G) \to \mathrm{End}(V)$ factorizes through $\mathrm{End}_{\mathrm{fin}}(V)$. Combined with the isomorphism $V \otimes V' \xrightarrow{\sim} \mathrm{End}_{\mathrm{fin}}(V)^{\mathrm{sm}}$, we get a homomorphism $\pi \colon \mathcal{H}(G) \to V \otimes V'$.

Proposition 19. *If V is a smooth admissible representation of G, then so is V'. Let V'' be the contragredient of V', then $V \to V''$ is an isomorphism of G-modules.*

6 missing content...

Proposition 20. *If G is a unimodular td-group, for all $\phi \in \mathcal{H}(G) \setminus \{0\}$, there exists an irreducible (π, V_π) such that $\pi(\phi) \neq 0$.*

We need to find $\phi' \in \mathcal{D}_c(G)$ such that $\phi * \phi'$ isn't nilpotent. We can take μ an invariant distribution such that $\mu(\mathbb{1}_K) \in \mathbb{Q}_{>0}$ for all $K \leq G$ compact open. From μ, we get $C_c^\infty(G) \cong \mathcal{H}(G)$. We can put a convolution on $C_c^\infty(G)$:

$$\phi * \phi' = \int_G \phi(h) \phi'(h^{-1} g) \, d\mu(h) \tag{42}$$

Take $\phi' = \overline{\check{\phi}}$. Then if $\phi \neq 0$,

$$\phi * \phi'(e_g) = \int_G \phi(h) \overline{\phi(h)} \, d\mu > 0 \tag{43}$$

Observe that $\overline{(\phi * \phi')^\vee} = \phi * \phi'$. The same argument gives $\phi * \phi' * \phi * \phi' \neq 0$. By induction, $(\phi * \phi')^{*2^n} \neq 0$, so $\phi * \phi'$ is not nilpotent.

7 Abelian td-groups

Since all irreducible representations are of dimension one, we are interested in particular in the characters. Let $\Omega(G) \coloneqq \{\text{smooth characters of } G\}$. $\mathcal{D}_c(G)$ should be continuous on $\Omega(G)$. We put the coarsest topology on $\Omega(G)$ such that for all $\xi \in \mathcal{D}_c(G)$, $\xi^{-1}(0) \cap \Omega(G)$ is closed.

Proposition 21. *For every compact open subgroup $K \leq G$, let $\Omega(G; K)$ be the subgroup of $\Omega(G)$ of characters $\xi\colon G \to \mathbb{C}^\times$ which are trivial on K. Then $\Omega(G; K)$ is open in $\Omega(G)$.*

Proof. Let $V_K \coloneqq \left\{ \xi \in \Omega(G) \;\middle|\; \xi(\chi) = 0, \forall \xi \in \mathcal{D}_c(G) * e_K \right\} = \ker e_K$, which is closed. We claim $\Omega(G; K) = \Omega(G) \setminus V_K$: if there were a $g \in K$ such that $r_g\chi \neq \chi$, this is equivalent to $x \notin \Omega(G; K)$. Then there exists $\xi \in \mathcal{D}_c(G)$ such that $\xi(r_g\chi) \neq \xi(\chi)$. ☐

8 Multiplicative groups

Let F be a non-archimedean local field with R its ring of integers with residue field $\mathbb{F}_q$. For example, $\mathbb{Q}_q$ for prime q. We have an exact sequence

$$0 \longrightarrow R^\times \longrightarrow F^\times \xrightarrow{\text{ord}} \mathbb{Z} \longrightarrow 0 \tag{44}$$

For fixed u_F, we have $F^\times \cong R^\times \times \mathbb{Z}$. $x = \arg(x) u_F^{\text{ord}(x)}$.

$$\chi(x) = \omega(\arg(x)) t^{\text{ord}(x)} \tag{45}$$

for some $t \in \mathbb{C}^\times$, and ω is a smooth character of $R^\times$. $R^\times$ is the maximal compact open subgroup of $F^\times$. Note that

$$\{\, 1 + u_F^n R \mid n \geq 1 \,\} = \left\{ x \in R \;\middle|\; |x-1|_q \leq q^{-n} \right\} \subset \mathbb{Q}_p \tag{46}$$

forms a neighbourhood basis of 1 in $R^\times$. Any smooth character ω of $R^\times$ will be trivial on some $1+u_F^nR$ (a compact open subgroup), so ω is of finite order and

$$\Omega(R^\times) = \varinjlim_n \Omega\big(R^\times/(1+u_F^nR)\big) \tag{47}$$

which is an infinite discrete torsion group.

- $\Omega(F^\times) = \Omega(R^\times)\times\mathbb{C}^\times$, which is a union of countably many copies of $\mathbb{C}^\times$. Each component of $\Omega(F^\times)$ is indexed by $\omega\in\Omega(R^\times)$ and written as $\Omega(F^\times,\omega)$. This is an algebraic variety!

Proposition 22. *There is an algebraic homomorphism*

$$\mathcal{D}_c(F^\times) \to \Gamma(\Omega(F^\times),0) = \prod_{\omega\in\Omega(R^\times)} \Gamma(\Omega(F^\times,\omega),0) \tag{48}$$

where $\Gamma(\Omega(F^\times,\omega),0) = \mathbb{C}[t,t^{-1}]$. *Moreover, if you restrict it to* $\mathcal{H}(F^\times)$, *you get an isomorphism to* $\Gamma_0(\Omega(F^\times,0))$, *the subalgebra of algebraic regular functions on* $\Omega(F^\times)$ *vanishing on all but finitely many components of* $\Omega(F^\times)$.

Proof. Over $\Omega(F^\times,\omega)$, we have a smooth representation $(\pi_\omega,\mathbb{C}[t^{\pm1}])$ defined by $\pi_\omega(x)\cdot\phi = \omega(\arg(x))t^{\mathrm{ord}(x)}\phi$ for all $\phi\in\mathbb{C}[t^{\pm1}]$. For every $\xi\in\mathcal{D}_\mathrm{c}(F^\times)$, and $\omega\in\Omega(R^\times)$, the action $\pi_\omega(\xi)$ on $\mathbb{C}[t^{\pm1}]$ is an endomorphism of $\mathbb{C}[t^{\pm1}]$-modules. Hence there exists a unique $\hat{\xi}_\omega\in\mathbb{C}[t^{\pm1}]$ representing this action. So the map $\xi\mapsto\hat{\xi} := (\hat{\xi}_\omega)_{\omega\in\Omega(R^\times)}$ is an algebraic homomorphism from $\mathcal{D}_\mathrm{c}(F^\times)$ to $\Gamma(\Omega(F^\times),0)$. For every $K\leq R^\times$ compact open and every $\xi\in_K *\mathcal{D}_\mathrm{c}()^\times)$, we have $\hat{\xi}_\omega = 0$ unless $\omega\in\Omega(R^\times/K)$. Hence $\mathcal{H}(R^\times)$ is sent to $\Gamma_0(\Omega(F^\times),0)$ by our homomorphism. For every smooth character $\omega\in\Omega(R^\times)$, we construct $e_\omega\in\mathcal{H}(R^\times)$ via $e_\omega := \omega\mu(R)^{-1}\mu$, where $\mu\in\mathcal{D}(F^\times)^{F^\times}$, and $\omega\colon F^\times\to\mathbb{C}$ is the extension by 0 of $\omega\colon\mathbb{R}^\times\to\mathbb{C}$. □

Claim 23. $e_\omega * e_{\omega'} = \delta_{\omega\omega'}e_\omega$

Proof. Observe:

$$\begin{aligned} e_\omega * e_{\omega'} &= \left(\mu(R)^{-2}\int_{R^\times}\omega(h)\omega'(h^{-1}\cdot)\,d\mu(h)\right)\mu \\ &= \left(\mu(R)^{-1}\int_{R^\times}\omega(h)\omega'(h^{-1})\,d\mu(h)\right)\omega'\mu(R)^{-1}\mu \end{aligned} \tag{49}$$

□

Claim 24.

$$\mathcal{H}(F^\times) = \bigoplus_{\omega\in\Omega(R^\times)} e_\omega * \mathcal{H}(F^\times) \tag{50}$$

The map in proposition 22 is always injective, but not necessarily surjective.
Definition. $\xi \in \mathcal{D}(G)$ is essentially compact if for all $\phi \in \mathcal{H}(G)$, $\xi * \phi \in \mathcal{H}(G)$. We denote it $\mathcal{D}_{\mathrm{ec}}(G)$.
Proposition 25. *We can extend equation* (48) *to* $\mathcal{D}_{ec}(G) \xrightarrow{\sim} \Gamma(\Omega(F^\times), 0)$.

8.1 Compact td-groups

Proposition 26. *Let G be compact. Every smooth representation of G is a union of finite-dimensional subrepresentations. In particular, irreducible smooth representations of G are finite-dimensional.*

Proof. Like "countable as ∞". □

Let (π, V) be an irreducible representation of G, and take $v \in V$,$v^* \in V^*$. Define m_{v,v^*} on G by

$$m_{v,v^*}(g) = \langle v, \pi^*(g)v^* \rangle = \langle \pi(g^{-1}), v^* \rangle \tag{51}$$

Note that for all g_1, $g_2 \in G$, $m_{\pi(g_1)v,\pi^*(g_2)v^*} = l_{g_1} r_{g_2} m_{v,v^*}$. Since v is smooth, $m_{v,v^*} \in C^\infty(G)^{(\mathrm{sm} l_G)}$. If $v^* \in V'$ ($= V^*$ if G is compact), then $m_{v,v^*} \in C^\infty(G)^{(\mathrm{sm} l_G \times r_G)}$. This defines a $G \times G$-equivariant map

$$m_\pi \colon V \otimes V' \to C^\infty(G)^{(\mathrm{sm} l_G \times r_G)} \tag{52}$$

Definition.

- An irreducible representation (π, V) of G is compact if $m_\pi(V \otimes V') \subseteq C_c^\infty(G)$ (this can't happen uness the centre Z of G is compact).
- An irrep (π, V) is compact modulo the centre (or cmc) if for all $v \otimes V' \otimes V'$, $m_{v\otimes v'}$ has support in ZC for some $C \subseteq G$ compact open.

Proposition 27. *Let (π, V) be a smooth rep of G. For all $v \in V$ and for all $K \leq G$ compact open, define*

$$\begin{aligned} \phi_{K,v} \colon G &\to V \\ g &\mapsto \pi(e_k * \delta_G * e_K)v \end{aligned} \tag{53}$$

If π is cmc, then $\operatorname{span}(\operatorname{im}\phi_{K,v})$ *is finite-dimensional.*

Proof. Suppose not. Then there exists $g_1, g_2, \dots \in G$ sich that $v_i \coloneqq \phi_{K,v}(g_i) \in V$ is linearly independent. So the open cosets ZKg_iK are distinct (i.e. disjoint), hence any compact $C \subseteq G$ will intersect finitely many. That is, $setg_i \nsubseteq ZC$. Take $v^* \in V^*$ such that $\langle v_i, v^* \rangle = 1$. Let $v' \coloneqq \pi^*(e_K)v* \in V'$ and $\langle v_i, v' \rangle = 1$ for all i. Then $m_{v,v'}(g_i) = 1$ for all i, so its support contains $\{g_i\}$. This is a contradiction. □

Proposition 28. *If π is cmc, then it is admissible.*

Proof. Note that $\varepsilon_K * \delta_G * e_K = \mathbb{1}_{KgK}$ generate $\mathcal{H}_K(G) = \left\{ f \in C_c^\infty(G) \mid f(KgK) = f(g), \forall g \right\}$ as a $\mathbb{C}$-vector space, so $\operatorname{span}(\operatorname{im} \phi_{K,v})$ is a $\mathcal{H}_K(G)$-invariant subspace of V^K. But V^K is irreducible as a representation of $\mathcal{H}_K(G)$ (since (π, V) is irreducible). Otherwise, $\pi(e_K)^{-1}(U) \leq V$. So $V^K = \operatorname{span}(\operatorname{im} \phi_{K,v})$. □

In particular, V' is irreducible (since $V'' \cong V$ is irreducible), hence $V \otimes V'$ is irreducible as a $G \times G$-representation.

Now assume (π, V) to be compact. Fix μ a Haar measure of G. So we have a map

$$\begin{aligned} \mu\colon C_c^\infty(G) &\to^{\cong} \mathcal{H}(G) \\ \phi &\mapsto \phi\mu \end{aligned} \tag{54}$$

which is $G \times G$-equivariant. Also, since π is admissible, $\pi(\mathcal{H}(G)) \subseteq \operatorname{End}_{\text{fin}}(V)^{\text{sm}} \cong V \otimes V'$. So $\pi \circ \mu \circ m_\pi \colon V \otimes V' \to V \otimes V'$ is a morphism of $G \times G$-representations. By Schur's Lemma, it is a multiplication by $c_\mu(\pi) \in \mathbb{C}$.

Proposition 29. *$C_\mu(\pi) \neq 0$.*

Proof. Take $(\pi', V_{pi'})$ an irreducible smooth re of G no isomorphic to π. Then $\pi' \circ \mu \circ m_\pi \colon V \otimes V' \to \operatorname{End}(V_{\pi'})$ is $G \times G$-equivariant. So for all $u \in V_{\pi'}$, the map

$$\begin{aligned} V \otimes V' &\to V_{\pi'} \\ w &\mapsto (\pi' \circ \mu \circ m_\pi(w))u \end{aligned} \tag{55}$$

is G-equivariant (acting on the left factor). But as a left G-module, $V \otimes V' \cong \bigoplus V$, since $\pi \ncong \pi'$, this map has to be 0. So for all $w \in V \otimes V'$, $\pi' \circ \mu \circ m_\pi(w) = 0$. Take $w \neq 0$ such that $\phi \coloneqq \mu \circ m_\pi(w) \in \mathcal{H}(G) \setminus \{0\}$ By the separation lemma, $\pi(\phi) \neq 0$. □

Remark 30. If G is compact, $c_\mu(\pi) = \dim(V)^{-1}\operatorname{vol}_\mu(G)$. Let $d_\mu(\pi) \coloneqq c_\mu(\pi)^{-1}$ (called the formal degree). Then

$$h_\pi \coloneqq d_\mu(\pi)\mu \circ m_\pi \colon V \otimes V' \to \mathcal{H}(G) \tag{56}$$

is a $G \times G$-equivariant section of π (h_π is not dependent on μ).

Proposition 31. *h_π is an injective homomorphism of algebras. Its image, $\mathcal{H}(G)_\pi$, is a two-sided ideal of $\mathcal{H}(G)$.*

Proof. Since $\pi' \circ h_\pi = 0$ if π' is irreducible and $\pi \not\cong \pi$, then for all irreducible π' distinct from π,

$$\pi'\big(h_\pi(\omega_1) * h_\pi(\omega_2)\big) = 0 = \pi'\big(h_\pi(\omega_1\omega_2)\big) \tag{57}$$

Also,

$$\pi\big(h_\pi(\omega_1) * h_\pi(\omega_2)\big) = \omega_1\omega_2 = \pi\big(h_\pi(\omega_1\omega_2)\big) \tag{58}$$

By the separation lemma, $\pi\big(h_\pi(\omega_1) * h_\pi(\omega_2)\big) = \pi\big(h_\pi(\omega_1\omega_2)\big)$.

Next, take for all $\omega \in V \otimes V'$ and for all $\phi \in \mathcal{H}9G)$, given π' irreducible and not equivalent to π, $\pi'\big(h_\pi(\omega) * \phi\big) = 0 = \pi'\big(h_\pi(\omega\phi)\big)$. Also, $\pi\big(h_\pi(\omega) * \phi\big) = \omega\pi(\phi) = \pi\big(h_\pi(\omega\phi)\big)$. By the separation lemma, $h_\pi(\omega) * \phi = h_\pi\big(\omega\pi(\phi)\big) \in \mathcal{H}(G)_\pi$. The same argument holds for $\phi * h_\pi(\omega)$. □

Let $\mathcal{H}(G)_\pi^\perp \coloneqq \ker\pi$. Then $\mathcal{H}(G) \coloneqq \mathcal{H}(G)_\pi \oplus \mathcal{H}(G)_\pi^\perp$ as $G \times G$-modules (the second factor being a two-sided ideal).

Let us now consider the case when G is compact. Then $e_\pi \coloneqq h_\pi(1_V)$ is a unit of $\mathcal{H}(G)_\pi$. Note that $1_V = \sum_{i=1}^n e_i \otimes {e_i}^*$ for e_i a basis. In fact, e_π are central idempotents of $\mathcal{H}(G)$ such that $e_\pi * e_{\pi'} = 0$ if $\pi \not\cong \pi'$.

Proposition 32. *Let G be a compact td-group. Then we have a morphism*

$$\bigoplus_{\pi \text{ irr.}} h_\pi \colon \bigoplus_{\pi \text{ irr.}} V_\pi \otimes {V_\pi}^* \to \mathcal{H}(G) \tag{59}$$

is an isomorphism of algebras.

Proof.

- (Injectivity): Suppose $\pi_1, \dots, \pi_n$ are non-isomorphic irreducible representations of G and $w_i \in V_i \otimes V_i^*$ such that $\sum_{i=1}^n h_{\pi_i}(w_i) = 0$. Then

$$0 = e_{\pi_j} * \sum_{i=1}^n h_{\pi_i}(w_i) = e_{\pi_j} * \sum_{i=1}^n e_{\pi_i} * h_{\pi_i}(w_i) = e_{\pi_j} * h_{\pi_j}(w_j) = h_{\pi_j}(w_j) \tag{60}$$

  So $w_j = 0$ for all j.

- (Surjectivity): For all $\phi \in \mathcal{H}(G)$, there exists $K < G$ compact open such that $e_K * \phi = \phi$. Taking K smaller, we assume it is normal. So unless $\pi \colon G \to \mathrm{End}(V_\pi)$ factors through the finite set G/K, then $e_\pi * e_K = H_\pi(\pi(e_K)) = 0$ (i.e. $e_\pi * \phi = e_\pi * e_K * \phi = 0$). So we can define $\phi' := \phi - \sum_{\pi \text{ irr}} h_\pi\big(\pi(\phi)\big)$. But then for any π' an irreducible representation of G, $\pi'(\phi') = \pi'(\phi) - \sum_{\pi \text{ irr}} \pi'\big(h_\pi\big(\pi(\phi)\big)\big) = \pi'(\phi) - \pi'(\phi) = 0$. By the separation lemma, $\phi' = 0$ and $\phi = \sum_{\pi \text{ irr.}} h_\pi\big(\pi(\phi)\big)$. □

Proposition 33 (Peter-Weyl)**.** *For G compact and (π', W) a smooth representation of G,*

$$W = \bigoplus_{\pi \ irr.} \pi'(e_\pi)W \tag{61}$$

Moreover, $\pi'(e_\pi)W \cong V_\pi^{\oplus k}$.

Proof. Since the e_π's are central, $\pi'(e_p i)W$ is a submodule of W. So let us show $\bigoplus_{\pi \text{ irr.}} \pi'(e_\pi)W \to W$ is an isomorphism.

- (Injectivity): $\sum_{i=1}^n \pi'(e_{\pi_i})w_i = 0$. Then

$$0 = \pi'(e_{\pi_j}) \sum_{i=1}^n \pi'(e_{\pi_i})w_i = \pi'(e_{\pi_j})w_j \tag{62}$$

- (Surjectivity): For all $w \in W$, there exist s$K < G$ compact open such that $\pi'(e_K)w = w$. But $e_K = \sum_{\pi \text{ irr.}} h_\pi\big(\pi(e_K)\big) = \sum_{\pi \text{ irr.}} e_\pi * e_K$. So

$$w = \pi'(e_K)w = \sum_{\pi \text{ irr.}} \pi'(e_\pi)\pi'(e_K)w = \sum_{\substack{\pi \text{ irr.} \\ e_\pi * e_K \neq 0}} \pi'(e_\pi)w \tag{63}$$

- ($\bigoplus$): We have $e_\pi \mathcal{H}(G) \otimes W \to \pi'(e_\pi)W$ G -equivariant and surjective where $g(\phi \otimes w) = (\delta_g * \phi) \otimes w)$. Since $\pi \colon e_\pi\mathcal{H}(G) = \mathcal{H}(G)_\pi \xrightarrow{\sim} V_\pi \otimes V_\pi^*$

is G-equivariant, (where G acts on the left factor of $V_\pi \otimes V_\pi^*$) As a left G-module, $V_\pi \otimes V_\pi^* \cong V_\pi^{\oplus k}$ Hence so is $e_\pi \mathcal{H}(G) \otimes W$. So $W \cong \sum V_\pi$ Since V_π is irreducible, $W \cong V_\pi^{\oplus k}$.

□