Quantum Groups

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Abstract

Quantum groups have a rich theory. Categorically they are well-behaved under the reversal of arrows, and algebraically they form an interesting generalization of both groups and Lie algebras. Additionally, they are a useful tool in knot theory, as much of their structure encodes various knot-theoretical properties. This summer project was devoted to understanding quantum groups (or by their other name, Hopf algebras) and bridging some of these connections.

1 Introduction

Here we will go over some basic definitions and properties of Hopf algebras.

Definition. An *algebra* is a vector space A over a field k together with an associative multiplication $m\colon A \otimes A \to A$, and a multiplicative unit $1 \in A$. In fact, this unit can be extended to a map $\eta\colon k \to A$ by linearly extending the assignment $\eta(1) = 1$.

We can express these axioms in terms of commutative diagrams. Denote by 1 the identity map on A. In particular,

$$\begin{array}{ccc} A^{\otimes 3} & \xrightarrow{m\otimes 1} & A^{\otimes 2} \\ {\scriptstyle 1\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A^{\otimes 2} & \xrightarrow{m} & A \end{array} \qquad (1) \qquad\qquad \begin{array}{ccc} A & \xrightarrow{1\otimes \eta} & A^{\otimes 2} \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m} \\ & & A \end{array} \qquad (2)$$

By taking the appropriate dual (for instance, if A is graded, then graded dual), we get the same diagrams, but with the arrows reversed. This leads us the notion of a coalgebra:

Definition. A *colagebra* is a vector space C over a field k with a *comultiplication* $\Delta\colon C \to C \otimes C$ which is *coassociative* (3) and a *counit*, which is a map $\epsilon\colon A \to k$ satisfying (4).

$$\begin{array}{ccc} C^{\otimes 3} & \xleftarrow{\Delta\otimes 1} & C^{\otimes 2} \\ {\scriptstyle 1\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ C^{\otimes 2} & \xleftarrow{\Delta} & C \end{array} \qquad (3) \qquad\qquad \begin{array}{ccc} C & \xleftarrow{1\otimes\epsilon} & C^{\otimes 2} \\ & {\scriptstyle 1}\nwarrow & \uparrow{\scriptstyle \Delta} \\ & & C \end{array} \qquad (4)$$

If an algebra A is also a coalgebra, it is natural to ask for compatibility between these two structures.

Definition. A *bialgebra* is an algebra (A, m, η) and a coalgebra (A, Δ, ϵ), such that one of the two equivalent conditions hold:

- Both Δ and ϵ are algebra maps.
- Both m and η are coalgebra maps.

Finally, given a bialgebra, we introduce an inverting operator of sorts, called the antipode:

Definition. A *Hopf algebra* is a bialgebra A with an *antipode* $S\colon A \to A$ such that

$$\begin{array}{ccc} A & \xrightarrow{\epsilon /\!\!/ \eta} & A \\ {\scriptstyle \Delta}\downarrow & & \uparrow{\scriptstyle m} \\ A \otimes A & \underset{1\otimes S}{\overset{S\otimes 1}{\rightrightarrows}} & A \otimes A \end{array} \qquad (5)$$

2 Examples

To see where Hopf algebras fit into what we already know, let us consider some classical examples:

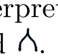
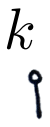
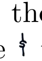
Example. For a (finite) group G, the group algebra $k[G]$ is a Hopf algebra with comultiplication $\Delta(g) = g \otimes g$, counit $\epsilon(g) = 1$, and antipode $S(g) = g^{-1}$ for all $g \in G$.

Example. Associated to a group (especially Lie or algebraic) we have a Lie algebra $\mathfrak{g}$. By "differentiating" the previous Hopf algebra, we find a Hopf algebra structure on $\mathfrak{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. That is, $\Delta(X) = 1 \otimes X + X \otimes 1$, $\epsilon(X) = 0$, and $S(X) = -X$ for all $X \in \mathfrak{g}$.

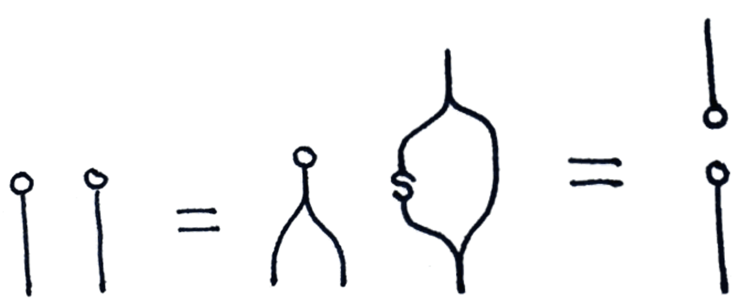
Example. Dual to the group algebra is the algebra $\hat{G}$ of functions $G \to k$ with pointwise multiplication. When G is finite, we have a natural correspondence $\hat{G} \otimes \hat{G} \cong (G \times G)^\wedge$, so the comultiplication becomes a map from univariate functions to bivariate functions. Given $f : G \to k$, we get $\Delta(f)(x, y) \coloneqq f(xy)$. The counit is $\epsilon(f) \equiv f(e)$, where $e \in G$ is the group identity, and $(Sf)(x) = f(x^{-1})$.

3 Graphical Calculus

Here we discuss some useful notation for expressing computations with Hopf algebras. The above definition of a Hopf algebra dealt with five fundamental maps, m, Δ, η, ϵ, and S. Each of these maps are distinguishable from each other by their source and target. From this observation, we can try to represent each of these functions more visually.

We will represent a vertical strand as the identity map on A. Two adjacent strands correspond to the tensor product of the identity map, i.e. $1_{A \otimes A}$. To represent multiplication, we will use the symbol . Reading from top to bottom, this is interpreted as a map from $A \otimes A$ to A. Similarly, comultiplication is denoted . By the natural identification $A \otimes k \cong A$, a map whose source is k should be written as if it had no source. Hence we will write the unit as , and likewise the counit as , indicating that they insert or remove a copy of A in a tensor power. Finally, we will use  to represent the antipode (i.e. a vertical line decorated with the letter S).

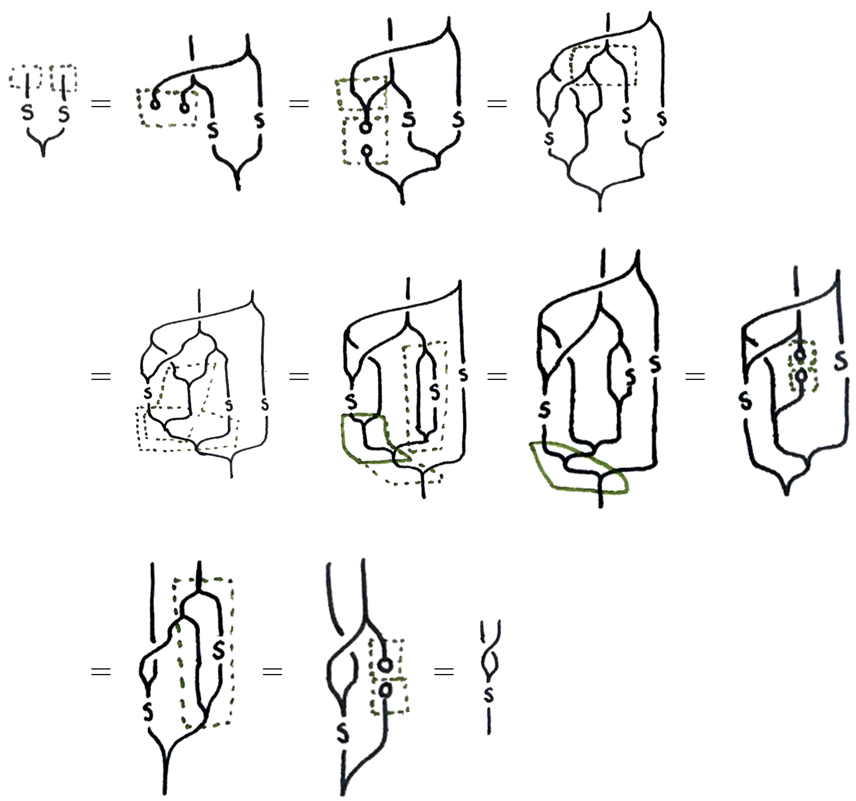
We are now in a position to express the above commutative diagrams with this new graphical calculus. For instance, the claim that the comultiplication is unital is expressed as



(a) $\Delta(1) = 1 \otimes 1$ (b) Equation (5)

Figure 1: Some identities expressed with the graphical calculus.

To see the usefulness of this calculus, consider the following proof that the antipode is an antihomomorphism (that is, $S(xy) = S(y)S(x)$):



One observes that any formula built from the axioms of a Hopf algebra has a “dual” formula obtained by reversing the arrows in the corresponding commutative diagram. In our diagrammatic language, this means that an equation of diagrams is true if and only if its reflection about the horizontal axis is true. It follows that one immediately concludes that S is also a coalgebra anticohomomorphism.

4 Further Structure

Here we discuss similar algebraic objects that one can study in this vein, in particular by adding or modifying the structure of a Hopf algebra.

4.1 Quasitriangular Hopf algebras

To motivate the next enrichment of structure, we introduce tangle diagrams.

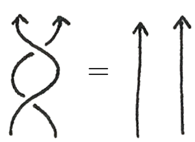
Definition. A *tangle* is a finite collection of disjoint embeddings of intervals into $\mathbb{H}^3$, considered up to endpoint-fixing homotopy.

For our purposes, tangle diagram is an encoding of a generic projection of such an embedding, namely a link diagram with open ends and no closed components. We will further consider *virtual* tangles, whereby the tangle diagram need not be planar.

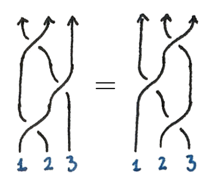
In order to encapsulate the notion of a crossing algebraically, we use the following notion:

Definition. A *quasitriangular* Hopf algebra H is a Hopf algebra with a distinguished element $\mathcal{R} \in H \otimes H$ which satisfies the properties required to make it compatible with tangle diagrams. That is, with $\mathcal{R} = \sum_i \mathcal{R}_i^1 \otimes \mathcal{R}_i^2$, we take $\mathcal{R}_{ij}$ to be the embedding of $\mathcal{R}$ into $H^{\otimes n}$ with the $\mathcal{R}_k^1$'s in the i-th component of $H^{\otimes n}$ and the $\mathcal{R}_k^2$'s in the j-th component of $H^{\otimes n}$. Then viewing $\mathcal{R}_{ij}$ as "the crossing of strand i over strand j" and multiplication as vertical concatenation of tangle diagrams, the statement of Reidemeister 2 (equation (7)) is that $\mathcal{R}$ is invertible, and Reidemeister 3 (equation (8)) is the Yang Baxter equation, i.e.

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{6}$$



(7)



(8)

4.2 Meta-Hopf algebras

One can loosen the definition of a Hopf algebra by not requiring A to be a vector space. Indeed, one can define Hopf algebra objects in any monoidal category. For instance, if one considers a Hopf algebra structure in $(\mathbf{Set}, \times)$ with the Cartesian product, then one recovers the definition of a group!

This definition can be further generalized, however. In our definition of a Hopf algebra, the only objects considered were tensor powers of the algebra: $k = A^{\otimes 0}$, A, $A \otimes A$, ... One can think of this as a collection of objects indexed by the natural numbers, $n \mapsto A^{\otimes n}$. Meta-Hopf algebras are defined by replacing the index set to be all of **FinSet**, and by replacing the algebras being indexed with any object. Now, given a finite set S, we have an associated "S-th tensor power" of A, denoted A_S. The operations of this space now must also specify which of the factor(s), labelled by elements of S, are being operated on, and the name of the new factor(s) produced by the product.

Example. One important example of a meta-Hopf algebra is the collection A of (framed) virtual tangle diagrams. A_S is the collection of tangle diagrams with strands labelled by elements of S, multiplication is given by joining the tip of the first strand to the tail of the second, and comultiplication is doubling of a strand along its framing. The unit is the insertion of a strand with no framing and no crossings with any other strand, and the counit corresponds to deletion of a strand. The antipode is simply reversal of the strand's orientation.

A morphism of meta-Hopf algebras with this one as its domain provides us with a tangle invariant that respects the operation of strand doubling.

5 Related Topics

5.1 Drinfel'd double

Also discussed over the summer were the *Drinfel'd double*, which is a process that takes a Hopf algebra H and produces a new, "doubled" algebra structure on $(H^*)^{\mathrm{op}} \otimes H$. This has a relationship to the study of w-tangles.

5.2 Quasi-Hopf algebras

When coassociativity fails in a Hopf algebra, one can recover this structure by considering quasi-Hopf algebras, wherein coassociativity holds up to conjugation by a specified element.

5.3 The "word problem" for Hopf algebras

The question of whether two words formed from the Hopf algebra operations are always equal (i.e. equal in every algebra) is a nontrivial task. Work by D. Bar-Natan and D. Thurston has built a connection of this problem to the study of triangularizations of 3-manifolds.

6 Acknowledgements and References

My main resources for this summer's project were *A Quantum Groups Primer* by Shahn Majid, the paper *Quantum Groups* by V.G. Drinfel'd, and meetings with Dror Bar-Natan. I would like to thank Dror for his insights and support over the summer; it has been an enlightening and exciting time.