Problem 1 (Spivak 22-1)**.** Prove that

$$\lim_{n\to\infty}\left[\frac{1}{n^{p+1}}\cdot\sum_{k=1}^{n}k^p\right]=\frac{1}{p+1} \tag{1}$$

Problem 2 (Spivak 22-2)**.** Compute the following limits:

1. $\lim\limits_{n\to\infty}\dfrac{n}{n+1}-\dfrac{n+1}{n}$
2. $\lim\limits_{n\to\infty} n-\sqrt{n+a}\sqrt{n+b}$
3. $\lim\limits_{n\to\infty}\dfrac{a^n-b^n}{a^n+b^n}$
4. $\lim\limits_{n\to\infty}\dfrac{2^{n^2}}{n!}$

Problem 3 (Spivak 22-3)**.** Consider the sequence

$$a_n=\left(\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},\frac{1}{7},\dots\right) \tag{2}$$

For which $\alpha\in\mathbb{R}$ does there exist a subsequence a_{n_k} which converges to α?

Problem 4 (Spivak 22-5)**.** Prove the sequence

$$\left(\sqrt{2},\sqrt{2\sqrt{2}},\sqrt{2\sqrt{2\sqrt{2}}},\dots\right) \tag{3}$$

converges and find its limit.

Problem 5.

1. Prove that a sequence converges if and only if all of its subsequences converge.
2. Prove that a Cauchy sequence converges if and only if it has a convergent subsequence (you only need one!).

Definition. Given a function $f\colon \mathbb{R} \to \mathbb{R}$, we define the <u>$n$-th compositional power</u> of f as

$$f^{\circ 0}(x) \coloneqq x$$
$$f^{\circ(n+1)}(x) \coloneqq f\big(f^{\circ n}(x)\big)$$

That is, $f^{\circ n}(x) = \underbrace{f(f(\cdots f}_{n}(x)\cdots))$.

Problem 6 (Spivak 22-20)**.** Let $x \in \mathbb{R}$ and f be continuous such that the sequence $a_n = f^{\circ n}(x)$ converges to ℓ. Prove that ℓ is a <u>fixed point</u> of f (that is, $f(\ell) = \ell$).

Problem 7 (*)**.** Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

- If it does, compute the limit.
- If it does not, given $M > 0$, find a good estimate for N to guarantee that $\sum_{n=1}^{N} \frac{1}{n} > M$.