Problem 1 (Spivak 20-1). Find the following Taylor polynomials

1. $f(x) = \mathbf{e}^{\mathbf{e}^x}$, degree 3 at 0.
2. $f(x) = \mathbf{e}^{\sin x}$, degree 3 at 0.
3. exp, degree n at 1.
4. $f(x) = x^5 + x^3 + x$, degree 4 at 0.
5. $f(x) = x^5 + x^3 + x$, degree 4 at 1.

Problem 2 (Spivak 20-7). Denote by a_n and b_n the coefficients of the Taylor polynomials at p of f and g, respectively. That is,

$$a_n = \frac{f^{(n)}(p)}{n!} \qquad b_n = \frac{g^{(n)}(p)}{n!} \tag{1}$$

Compute the coefficients of the Taylor polynomials (at p) of the following functions in terms of a_n and b_n:

1. $f + g$
2. fg
3. f'
4. $h(x) = \int_p^x f$
5. $s(x) = \int_0^x f$

Problem 3 (Spivak 20-11). Use Taylor's theorem to compute the following limits *without* using L'Hôpital's rule. *Hint:* Expand the numerator and denominator separately as Taylor polynomials

1. $\displaystyle\lim_{x\to 0} \frac{\mathbf{e}^x - 1 - x - \frac{1}{2}x^2}{x - \sin x}$
2. $\displaystyle\lim_{x\to 0} \frac{\frac{\mathbf{e}^x}{1+x} - 1 - \frac{1}{2}x^2}{x - \sin x}$ (*Hint:* Is there a nice expression for $\frac{1}{1+x}$?)
3. $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$
4. $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{\sin(x^2)}\right)$

Problem 4 (Spivak 20-6)**.** Here is a formula which is sometimes true:

$$\arctan x + \arctan y = \arctan\left(\frac{x+1}{1-xy}\right) \tag{2}$$

1. Prove that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \tag{3}$$

2. Use the second expression to prove $\pi = 3.14159\ldots$.